On the Suppression of Flutter in the Keldysh Model

plenary lecture

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Content

M. Keldysh: suppression of flutter

Mstislav V. Keldysh (1911-1978)

1931-1946: TsAGI

1942: The State Stalin Prize "**for scientific works on the prevention of aircraft destruction**"

1961-1975: President of the Academy of Sciences of the USSR

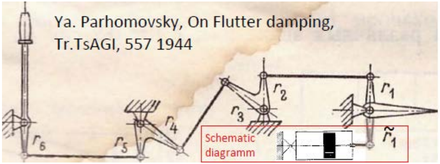
F. Lanchester (UK), Torsional vibrations of the tail of an aeroplane, **1916**: why Handley Page 0/400 **biplane bomber** had experienced **violent oscillations** of the fuselage and tail

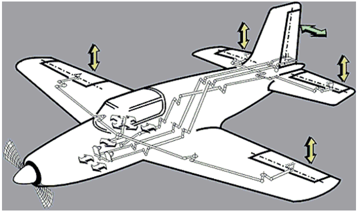
Flutter is a self-excited oscillation, often destructive, wherein energy is absorbed from the airstream; a complex phenomenon that must in general be completely eliminated by design or prevented from occurring within the flight envelope

Ya. Parhomovsky, L. Popov (TsAGI): about **150 crashes** of new models of aircrafts **caused by flutter in German aviation** during **1935-1943**
[Keldysh's research in TsAGI on the self-oscillations of aircraft structures, Tr. TsAGI, II(1), 1971]

M. Keldysh, 1944: suppression of flutter in airfoil with aileron

M. Keldysh, On dampers with a nonlinear characteristic, Tr. TsAGI 557 (1944) 26-37





Flutter suppression by hydraulic damper: resistance $= -f(\dot{x}) = -\lambda\dot{x} + (\Phi + \kappa\dot{x}^2)\mathrm{sign}(\dot{x})$

One degree of freedom:

$$J\ddot{x} + kx = h\dot{x} - f(\dot{x})$$

Keldysh's conditions by describing function method (DFM, harmonic balance):
if $h < \lambda + 2.08\sqrt{\Phi\kappa}$, then all trajectories converge to the equilibria segment;
if $h > \lambda + 2.08\sqrt{\Phi\kappa}$, then there are two limit cycles $\approx a_\pm \cos(\omega t)$ with amplitudes

$$a_\pm(\mu) = \frac{3}{8\kappa}\sqrt{\frac{J}{k}}\left(\pi(\mu) \pm \sqrt{\pi^2(\mu)^2 - \frac{32}{3}\kappa\Phi}\right), \quad \mu = -h + \lambda.$$

$$x \to \frac{J}{\kappa}x, t = t\frac{\sqrt{J}}{\sqrt{k}} \Rightarrow J = 1, k = 1, \kappa = 1.$$

Instability in large and oscillations: recent accidents

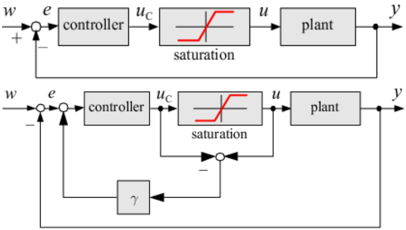
Pilot-Induced Oscillation – sustained or uncontrollable oscillations resulting from efforts of the pilot to control with high precision linked with actuator magnitude and rate saturation (Byushgens G.S., TsAGI)

Accidents: crash of YF-22 (Lockheed/Boeing) 1992; crash of JAS-39 Gripen (SAAB) 1993.

Windup: oscillations with increasing amplitude

Actuator saturations may lead to controller windup, which dramatically changes the overall closed-loop system performance.

Antiwindup – a scheme to avoid windup in control systems with saturation.



Development of methods of analysis

M. Keldysh had only the Describing Function Method (well developed by 1944). However, DFM is only an approximate method and may lead to wrong conclusions about the existence of periodic solutions.

M. Keldysh wrote (1944): " *We do not give a rigorous mathematical proof ..., we construct a number of conclusions on intuitive considerations* ..."

Nowadays we can apply rigorous analytical and reliable numerical methods, which have been developed during last 74 years (1944-2018):

1) Theory of differential equations with discontinuous right-hand sides and theory of differential inclusions;

2) Direct Lyapunov method and frequency methods;

3) Numerical algorithms for solving differential inclusions.

G. Leonov, N. Kuznetsov (2018) On the suppression of flutter in the Keldysh model, Doklady Physics

Study of dynamics in discontinuous models: practice & theory

Mechanical motivation and Mathematical theory of
Discontinuous Dynamical Equations and Differential Inclusion

French school: C.-A. de Coulomb (1781, friction),
P. Penleve (1895, rigid-body dynamics with both contact friction and Coulomb friction is inconsistent),
A. Marchaud (1934, contingent differential equations)

Poland school: S. Zaremba (1936, paratingent differential equations).
T. Wazewski (1961, solutions of the contingent differential equation are absolute continuous functions and, thus, derivative exists almost everywhere)

Russian school: A. Andronov (1937, point transformation methods for the construction of solutions of two-dimensional DDE, sliding motions)
A. Filippov (1960, consideration of DDE as DI, mathematical theory of DI based on the absolute continuous solutions)

Numerical computation:
M. Aizerman and E. Pyatnitskiy (1974, Approximation of DDE by continuous ODEs)
P. Piiroinen, Yu. Kuznetsov (2008, Event-driven method to simulate DI with accurate computing of sliding motions)

DI: $\dot{x} \in F(x)$, $\dot{x}(t)$ exists for almost all t, $F(x)$ is a convex set for each $x \in \mathbb{R}^n$

Global stability criteria for Differential Inclusions

1944: A. Lurie, V. Postnikov
Lyapunov functions in the form: quadratic form + integral of nonlinearity.

1949: M. Aizerman, **1957**: R. Kalman
Conjectures on the absolute stability of the control systems with one stationary point and nonlinearities from the sector of linear stability.
Describing Function Method (harmonic balance) claims the validity of the conjectures, however various counterexamples have been constructed.

1959-1963: M. Popov, V. Yakubovich, R. Kalman
Frequency conditions for the Lyapunov function existence.

1964-1996: A. Gelig, G. Leonov, V. Yakubovich
Frequency conditions of global stability for the differential inclusion.

✓ A.Kh. Gelig and G.A. Leonov and V.A. Yakubovich, Stability of Nonlinear Systems with Nonunique Equilibrium (in Russian), Nauka, 1978 [English transl: Stability of stationary sets in control systems with discontinuous nonlinearities, World Scientific, 2004]
✓ G.A. Leonov, D.V. Ponomarenko, V.B. Smirnova, Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications, World Scientific, 1996

New global stability criteria for Differential Inclusions

G. Leonov, N. Kuznetsov, On the suppression of flutter in the Keldysh model, Doklady Physics, **2018**

DI: $\dot{u} \in Pu + q\hat{\varphi}(r^*u), \quad u, q, r \in \mathbb{R}^n, \quad \hat{\varphi}(\sigma) = \begin{cases} \varphi(\sigma) & \sigma \neq 0, \\ [-\Phi, \Phi] & \sigma = 0, \end{cases}, \quad \varphi : \mathbb{R} \to \mathbb{R}$

$W(s) = r^*(P - Is)^{-1}q = \frac{sm(s)}{n(s)}$, (P, q) - controllable, (P, r) - observable; roots of $n(s)$ - non-positive real parts, $m(s)$ - negative real parts;

Lemma (GLYa, 1978). Suppose that 1) function $V(u(t))$ does not increase in t for any solution; 2) if for a bounded solution $u(t)$ the equality $V(u(t)) \equiv \text{const}$ holds for $0 \leq t < \infty$, then the solution $u(t)$ is a stationary vector; 3) $V(u) \to \infty$ for $|u| \to +\infty$ then any solution of DI tends to the stationary set.

Theorem 1. If $\varphi(\sigma)\sigma > 0$, $\forall \sigma \neq 0$ and $\text{Re}W(i\omega) \geq 0, \forall \omega \in \mathbb{R}$, then any solution of DI tends to the stationary set (and corresponding Lyapunov function exists).

Theorem 2. If $\varphi(\sigma)\sigma \geq \delta_e \sigma^2$ for $\delta_e \geq 0$; $\varphi'(\sigma) > -\delta$, $\sigma \neq 0$, for $\delta \geq 0$; there are $\tau \geq 0$ and θ: $\text{Re}(\omega^2 + \theta i\omega + \tau)W(i\omega) - \delta|W(i\omega)|^2\omega^2 + \tau\delta_e|W(i\omega)| \geq 0, \forall \omega \in \mathbb{R}$, then any solution of DI tends to the stationary set.

Keldysh problem (one degree of freedom): rigorous analysis

DI: $J\ddot{x} + kx + \mu\dot{x} \in -\hat{\varphi}(\dot{x}), \quad \mu = -h + \lambda$
$\varphi(\dot{x}) = (\Phi + \kappa\dot{x}^2)\mathrm{sign}(\dot{x})$

$$\hat{\varphi}(\dot{x}) = \begin{cases} \varphi(\dot{x}) & \dot{x} \neq 0 \\ [-\Phi, \Phi] & \dot{x} = 0 \end{cases}$$

Theorem 3. DI is dissipative in the sense of Levinson, i.e. possesses a global absorbing set.

Proof: Comparison systems and Lyapunov functions

Theorem 4. If $-\mathbf{2}\sqrt{\mathbf{\Phi}\kappa} < \mu$, then any solution of DI converges to the stationary segment $[-\Phi, \Phi]$.

Proof: Lyapunov-type theory for DI: $V(x,\dot{x}) = \frac{1}{2}(J\dot{x}^2 + kx^2)$, $\dot{V}(x,\dot{x}) = -\mu\dot{x}^2 - \dot{x}\varphi(\dot{x})$

Keldysh by Describing Function Method (DFM): $-\mathbf{2.08}\sqrt{\mathbf{\Phi}\kappa} < \mu$ all solutions converge to the stationary segment $[-\Phi, \Phi]$.

Global stability: Keldysh's estimate is close to the rigorous analytical estimate

Keldysh problem (one degree of freedom): harmonic balance

DI: $\ddot{x}+x\in-\mu\dot{x}-(\Phi+\kappa\dot{x}^2)\mathrm{sign}(\dot{x})$. The bifurcation of collision of limit cycle and stationary segment under Keldysh's condition: $\mu<-2.08\sqrt{\Phi}=-\delta_K \Rightarrow$ there are two limit cycles: $\approx a_\pm\cos(\omega t)$, $a_\pm(\mu)=-\frac{3\pi}{16}\mu\pm\frac{3\pi}{16}\sqrt{\mu^2-\delta_K^2}$ and the inner cycle is the boundary of stability domain of the stationary segment $[-\Phi,\Phi]$

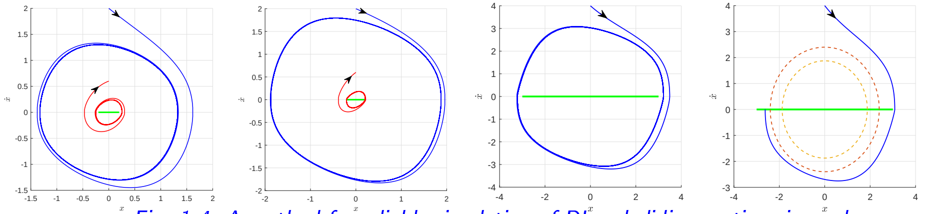


Fig. 1-4. A method for reliable simulation of DI and sliding motions is used

$\Phi=\frac{1}{5}$ **1)** $\mu=-1.3967\delta_K$: $a_+(\mu)\gg a_-(\mu)>\Phi$ **2)** $\mu=-1.7847\delta_K$: $a_+(\mu)\gg\Phi>a_-(\mu)$
Outer trajectory winds onto stable limit cycle, inner trajectory unwinds from unstable limit cycle and winds onto the stable limit cycle (hidden attractor).

$\Phi=3$ **3)** $\mu=-1.0713\delta_K$: $a_+(\mu)\gtrsim\Phi>a_-(\mu)$. Outer trajectory winds onto stable limit cycle, inner unstable limit cycle is not revealed numerically (due to stiffness).
4) $\mu=-1.0076\delta_K$: $\Phi\gtrsim a_+(\mu)>a_-(\mu)$: both limit cycles have disappeared.

Stability domain: Keldysh's estimate may be far from the numerical estimates

Keldysh problem (two degrees of freedom): rigorous analysis

DDE:
$$c_{11}\ddot{x}+c_{12}\ddot{y}+b_{11}\dot{x}+b_{12}\dot{y}+a_{11}x+a_{12}y=0, \quad \mu=b_{22}+\lambda$$
$$c_{21}\ddot{x}+c_{22}\ddot{y}+b_{21}\dot{x}+(b_{22}+h)\dot{y}+a_{21}x+a_{22}y=(h-\lambda)\dot{y}-(\Phi+\kappa\dot{y}^2)\mathrm{sign}(\dot{y})$$

Consider a minimal $h=h(b_{22})<\lambda$ such that the linear system $(\Phi=\kappa=0)$ is stable

$$\dot{u}=Pu+q\varphi(r^*u), \quad P=Q^{-1}\widetilde{P},\ q=Q^{-1}\widetilde{q}, \quad \varphi(r^*u)=(\Phi+\kappa(r^*u)^2)\mathrm{sign}(r^*u)$$

$$Q=\begin{pmatrix}1&0&0&0\\0&c_{11}&0&c_{12}\\0&0&1&0\\0&c_{21}&0&c_{22}\end{pmatrix}\widetilde{P}=\begin{pmatrix}0&1&0&0\\-a_{11}&-b_{11}&-a_{12}&-b_{12}\\0&0&0&1\\-a_{21}&-b_{21}&-a_{22}&-\mu\end{pmatrix}u=\begin{pmatrix}x\\z\\y\\v\end{pmatrix}\widetilde{q}=\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}r=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\dot{u}\in Pu+q\hat{\varphi}(r^*u)=(P+\delta qr^*)u+q(\hat{\varphi}(r^*u)-\delta r^*u)=P_\delta u+q\hat{\varphi}_\delta(r^*u)$$

DI:
$$\hat{\varphi}_\delta(\sigma)=\hat{\varphi}(\sigma)-\delta\sigma=\begin{cases}\varphi_\delta(\sigma)=(\Phi+\kappa(\sigma)^2)\mathrm{sign}(\sigma)-\delta\sigma & \sigma\neq 0\\ [-\Phi,\Phi] & \sigma=0\end{cases}$$

$$W(s)=r^*(P-Is)^{-1}q=\frac{s(c_{11}s^2+b_{11}s+a_{11})}{(c_{22}s^2+\mu s+a_{22})(c_{11}s^2+b_{11}s+a_{11})-(c_{21}s^2+b_{21}s+a_{21})(c_{12}s^2+b_{12}s+a_{12})}$$

Frequency methods allow us to estimate μ via parameters effectively:
$$\mathrm{Re}W(i\omega)^{-1}=\mu-\mathrm{Re}W_1(i\omega)=\mu-\frac{h_4(a_{ij},b_{ij},c_{ij})\omega^4+h_2(a_{ij},b_{ij},c_{ij})\omega^2+h_0(a_{ij},b_{ij},c_{ij})}{(a_{11}-c_{11}\omega^2)^2+b_{11}^2\omega^2}$$

Keldysh problem (two degrees of freedom): example

Example [Chen et al., Commun Nonlinear Sci Numer Simulat, 2012]:
$a_{11} = 2844.4, a_{12} = 344.6301, a_{21} = 0, a_{22} = 24.0556, b_{11} = 45.2997, b_{12} = 0, b_{21} = 0.5852, b_{22} = 0.0360, c_{11} = 15.5700, c_{12} = 0.4730, c_{21} = 0.4730, c_{22} = 0.0606.$
Matrix $P = P(\mu) = Q^{-1}\widetilde{P}(\mu)$ with $\mu = b_{22}$ has eigenvalues $\{-5.6754 \pm 16.4890i, 3.5715 \pm 17.3156i\}$. Minimal $\mu = \mu_c$ such that liner system is stable: $\mu_c = b_{22} + h = b_{22} + 2.3663...$ (i.e. $P(\mu)$ c $\mu > \mu_c = b_{22} + h$ has eigenvalues with negative real parts; for $\mu = \mu_c$ eigenvalues $\{-44.31, -11.07, \pm 13.91\, i\}$).
Consider $\kappa = 1/2$, $\Phi = 1/2$, therefore $\delta_0 = 2\sqrt{\Phi} = 1$, $\delta_K = \frac{8}{\pi\sqrt{6}} 2\sqrt{\Phi} \approx 1.04$.

All trajectories of tend to the stationary segment:

a) $\lambda > 1.68$ by Theorem 1;
b) $\lambda > 1.65$ by Theorem 2 with $\theta = -36.53$, $\tau = 500$, $\delta = 0.72$, $\delta_e = 0.28$;

c) $\lambda > 1.33$ by Keldysh's conditions (DFM);

d) $\lambda \in (1.34, 1.36]$ by numerical analysis;

For $\lambda \in [1.33, 1.34]$ limit cycles can be found numerically in the model, while Keldysh's condition declares that all trajectories tend to the stationary segment.

Conclusions

Lauvdal et al., Stabilization of integrator chains in the presence of magnitude and rate saturations; a gain scheduling approach, *CDC, 1997*: **Since stability in simulations does not imply stability of the physical control system** *(an example is the crash of the YF22)* **stronger theoretical understanding is required.**

✓ G.A. Leonov, N.V. Kuznetsov, On the suppression of flutter in the Keldysh model, **Doklady Physics**, **2018**
✓ B.R. Andrievsky, N.V. Kuznetsov, G.A. Leonov, Methods for suppressing nonlinear oscillations in astatic auto-piloted aircraft control systems, **J. of Computer and Systems Sciences Int.**, 56(3), **2017**, 455-470
✓ B. Andrievsky, K. Kravchuk, N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, Hidden oscillations in the closed-loop aircraft-pilot system and their prevention, **IFAC-PapersOnLine**, 49(14), **2016**, 30-35
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✓ A. Pogromsky, A. Matveev, B. Andrievsky, G. Leonov, N. Kuznetsov, A non-quadratic criterion for stability of forced oscillations and its application to flight control, **European Control Conf.**, **2013**, IEEE, art. 6669353
✓ B.R. Andrievsky, N.V. Kuznetsov, G.A. Leonov, A.Yu. Pogromsky, Hidden Oscillations in Aircraft Flight Control System with Input Saturation, **IFAC Proceedings Volumes**, 5(1), **2013**, 75-79
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✓ G.A. Leonov, B.R. Andrievskii, N.V. Kuznetsov, A.Yu. Pogromskii, Aircraft control with Anti-Windup compensation, **Differential equations**, 48(13), **2012**, 1700-1720
✓ G.A. Leonov, N.V. Kuznetsov, A.Yu. Pogromskii, Stability domain analysis of an antiwindup control system for an unstable object, **Doklady Mathematics**, 86(1), **2012**, 587-590

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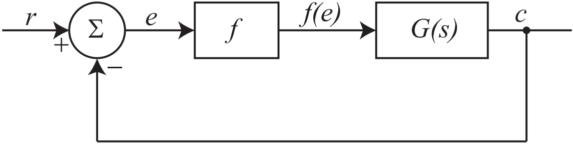
Questions

Thank you for your attention!

Kalman conjectures, 1957

[1] Kalman R. E., Physical and Mathematical mechanisms of instability in nonlinear automatic control systems, Transactions of ASME, 79(3), 1957, 553-566



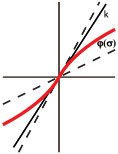


Fig.1. Nonlinear control system. $G(s)$ is a linear transfer function, $f(e)$ is a single-valued, continuous, and differentiable

In 1957 R.E. Kalman wrote [1]: " If $f(e)$ in Fig. 1 is replaced by constants K corresponding to all possible values of $f'(e)$, and it is found that the closed-loop system is stable for all such K, then it is intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point."

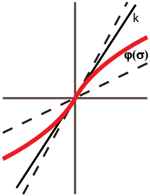
Engineering intuition: the statement is true for models in $\mathbb{R}^{1,2,3}$!

G.A. Leonov, N.V. Kuznetsov, Algorithms for Searching for Hidden Oscillations in the Aizerman and Kalman Problems, Doklady Mathematics, 84(1), **2011**, 475-481

In 2013 R.E. Kalman wrote: "*I was far too young and lacking technical mathematical knowledge to go more deeply into the matter.*"

M.Aizerman & R. Kalman problems and harmonic balance

if $\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k\in(k_1,k_2)$: $\mathbf{z}(t,\mathbf{z}_0)\to 0\ \forall \mathbf{z}_0$, then
is $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{b}\varphi(\sigma)$, $\sigma=\mathbf{c}^*\mathbf{x}$, $\varphi(0)=0$, $k_1<\varphi(\sigma)/\sigma<k_2$: $\mathbf{x}(t,\mathbf{x}_0)\to 0\ \forall\mathbf{x}_0$?



$1949: \ k_1 < \varphi(\sigma)/\sigma < k_2$

$1957: \ k_1 < \varphi'(\sigma) < k_2$

Describing Function Method: the system is monostable (e.g. there are no any periodic solutions) in the case of Aizerman's and Kalman's conditions.

In general, the conjectures are not true: Aizerman's in $\mathbb{R}^2$ and Kalman's in $\mathbb{R}^4$. Counterexamples: the only equilibrium, which is stable, coexist with a stable oscillations (hidden attractor) which basin of attraction is often small.

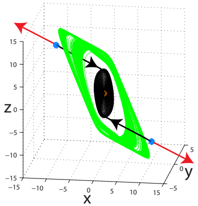
Survey: Leonov G.A.,Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, Int. J. Bif. and Chaos, 23(1), 2013, art. no. 1330002, doi: 10.1142/S0218127413300024

Multistability and hidden attractors

An attractor is called a *self-excited attractor* if its basin of attraction intersects with small neighborhood of an equilibrium, otherwise it is called a *hidden attractor*.

e.g. hidden attractors in the systems without equilibria, or with the only stable equilibrium (special case of multistability & attractors coexistence)

standard computational procedure: [1. to find equilibria; 2. trajectory, starting from a point of unstable manifold in a neighborhood of unstable equilibrium, after transient process reaches an self-excited oscillation and localizes it]
allows to find self-excited but not hidden oscillations.



How to predict the existence of a hidden attractor and choose an initial point in its attraction domain for the visualization?

Surveys: ✓ Leonov G.A.,Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, Int. J. Bif. and Chaos, 23(1), 2013, art. no. 1330002, doi: 10.1142/S0218127413300024
✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion, European Phys. J. Special Topics, 224, 2015, 1421-1458

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His interests are now in dynamical systems stability and oscillations, chaos, phase-locked loop, and nonlinear control systems.

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