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Numerical justification of Leonov conjecture on Lyapunov dimension of Rossler attractor

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Abstract. Exact Lyapunov dimension of attractors of many classical chaotic systems (such as Lorenz, Henon, and Chirikov systems) is obtained. While exact Lyapunov dimension for Rössler system is not known, G.A. Leonov formulated the following conjecture: *Lyapunov dimension of Rössler attractor is equal to local Lyapunov dimension in one of its stationary points.* In the present work Leonov's conjecture on Lyapunov dimension of various Rössler systems with standard parameters is checked numerically.

Keywords: Rössler system, Lyapunov dimension, strange attractor, self-excited and hidden attractor, Lyapunov exponent, chaos, Perron effects, Leonov's conjecture

1 Introduction

Lyapunov exponents (LEs) play an important role in the description of dynamical systems behavior. They were introduced by A.M. Lyapunov Lyapunov [1892] for the analysis of stability by the first approximation for *regular* time-varying linearizations, where the negativeness of the largest Lyapunov exponent indicates stability. Much later, in 1940s, N.G. Chetaev tried to prove that for *regular* time-varying linearizations, a positive Lyapunov exponent indicates instability in the sense of Lyapunov, but a gap in his proof was revealed and filled recently for more weak definition of instability Leonov and Kuznetsov [2007]). Since there are no general methods for checking regularity of linearization and there are known Perron effects Kuznetsov and Leonov [2005a,b,c]; Leonov and Kuznetsov [2007] of sign inversion of the largest Lyapunov exponent for nonregular time-varying linearizations, the computation of Lyapunov exponents for linearization of nonlinear autonomous system along nonstationary trajectories is widely used for investigation of chaos. In this case the positiveness of the largest Lyapunov exponent is often regarded as the indication of chaotic behavior in the considered nonlinear system. The various methods, used for the numerical computation of Lyapunov exponents, are described, e.g., in Benettin et al. [1980a,b]; Shimada and Nagashima [1979]; Wolf et al. [1985].

Nowadays various characteristics of attractors of dynamical systems (information dimension, metric entropy etc) are studied based on Lyapunov exponents computation. In particular, J.L. Kaplan and J.A. Yorke defined a quantity they called *Lyapunov dimension* and conjectured that it was equal to information dimension Kaplan and Yorke [1979].

In the work Leonov [2012] G.A. Leonov considered exact formulas of Lyapunov dimension of Lorenz, Henon, and Chirikov attractors. By analogy with the results for these attractors he conjectured that Lyapunov dimension of Rössler attractor[2] is determined by a stationary point belonging to this attractor.

In the present paper Leonov's conjecture is checked numerically and it is demonstrated that this conjecture is true for three different types of Rössler systems. These three-dimensional systems are simplest and, in a sense, minimal models for continuous-time chaos. They have only a single nonlinear quadratic term and they

[1] PDF slides http://www.math.spbu.ru/user/nk/PDF/Lyapunov-exponent-Sign-inversion-Perron-effects-Chaos.pdf

[2] Following Broer et al. [1991]; Leonov [2008], an attractor is a bounded, closed, invariant, attracting subset of phase space of dynamical system. Since for the considered Rössler systems there are no analytical estimations of localization of their attractors, it is not feasible to check their boundness and closedness. Usually by Rössler attractor one means an attracting set obtained as a result of numerical experiments Rossler [1976, 1979].

generate chaotic attractors with a single "leaf" (in contrast to Lorenz attractor). Rössler systems arose as simplified prototypes of some chemical reactions while Otto Rössler researched different types of chaos in chemical kinetics.

2 Problem statement

2.1 Rössler systems

Consider the following three-dimensional Rössler systems Rossler [1976, 1979]

$$\textbf{(1.1)}\begin{cases}\dot u = -y - z\\ \dot y = u\\ \dot z = a(y - y^2) - bz\end{cases} \qquad \textbf{(1.2)}\begin{cases}\dot u = -y - z\\ \dot y = u + ay\\ \dot z = b - cz + uz\end{cases} \qquad \textbf{(1.3)}\begin{cases}\dot u = -y - z\\ \dot y = u + ay\\ \dot z = bu - cz + uz\end{cases} \tag{2.1}$$

with the corresponding standard parameters

$$\begin{aligned}&(1.1): a = 0,386;\ b = 0,2;\\ &(1.2): a = 0,2;\ b = 0,2;\ c = 5,7;\\ &(1.3): a = 0,36;\ b = 0,4;\ c = 4,5.\end{aligned} \tag{2.2}$$

In the phase spaces of these systems, for parameters (2.2) there exist chaotic attractors and the corresponding stationary points

$$\begin{aligned}&x_0 = (0,\ 0,\ 0) \quad \text{for systems (1.1) and (1.3)},\\ &x_0 = \left(\frac{c - \sqrt{c^2 - 4ab}}{2}, -\frac{c - \sqrt{c^2 - 4ab}}{2a}, \frac{c - \sqrt{c^2 - 4ab}}{2a}\right) \text{ for system (1.2)}\end{aligned} \tag{2.3}$$

are located in the middle of these attractors Rossler [1976, 1979].

2.2 Lyapunov dimension

Consider a topological characteristic — a *local Lyapunov dimension* of the point x_0 in the phase space U of dynamical system, which is associated with the Lyapunov spectrum $\lambda_1(x_0) \geq \ldots \geq \lambda_n(x_0)$ and is defined by formula

$$\dim_L x_0 = j + \frac{\lambda_1(x_0) + \ldots + \lambda_j(x_0)}{|\lambda_{j+1}(x_0)|}. \tag{2.4}$$

Here $j \in [1, n]$ is the smallest natural number m such that

$$\lambda_1(x_0) + \ldots + \lambda_{m+1}(x_0) < 0, \quad \lambda_{m+1}(x_0) < 0, \quad \frac{\lambda_1(x_0) + \ldots + \lambda_m(x_0)}{|\lambda_{m+1}(x_0)|} < 1.$$

Lyapunov dimension of invariant set $B \subset U$ of dynamical system is defined by the relation

$$\dim_L B = \sup_{x \in B} \dim_L x. \tag{2.5}$$

The properties of Lyapunov dimension are considered in details in the works Pesin [1988]; Temam [1993]; Boichenko et al. [2005]. In particular, it is proved that Lyapunov dimension is an upper bound for Hausdorff and fractal dimensions.

2.3 Leonov's conjecture

For Lorenz, Henon, and Chirikov systems a problem of computation of Lyapunov dimension of their attractors is solved in Leonov and Lyashko [1997]; Leonov [1998]; Boichenko et al. [1998]; Boichenko and Leonov [2000]; Leonov et al. [2011a,b]. In these works it is obtained analytically exact Lyapunov dimension of attractors of these systems and in Leonov [2012] it is given estimates of Lyapunov dimension of attractor of Rössler system (1.1). Based on these results, G.A. Leonov formulated the following

Conjecture. *If a stationary point* x_0 *is embedded in attractor* A *of Rössler systems* (2.1)*, then*

$$\dim_L A = \dim_L x_0.$$

In order to verify this conjecture for attractors of systems (2.1) with parameters (2.2) and stationary points (2.3), in the present work it is developed a special numerical procedure described below. Note that this procedure can be applied similarly to various modifications of Rössler system of higher orders (see, e.g., Rossler [1979]; Szczepaniak and Macek [2008]; Li [2008]).

3 Numerical justification of Leonov's conjecture

3.1 Lyapunov spectrum computation algorithm

To verify the conjecture, it is used an approach to the computation of Lyapunov spectrum, suggested in the works Benettin et al. [1980a,b]. In Wolf et al. [1985] this approach was adapted to computer realization. This method is an iterative process and is a variation of standard QR algorithm for computation of eigenvalues and eigenvectors Golub and van Loan [1996]. It is based on the following definitions and statements.

Consider system (2.1) in general form

$$\dot{x} = F(x), \tag{3.6}$$

where $x(t) \in \mathbb{R}^n$ for any $t \in \mathbb{R}$, $F : U \to \mathbb{R}^n$ is C^r-smooth function ($r \geq 1$) on the open set $U \subset \mathbb{R}^n$.

Denote by $A(t) = T_x F\left(f(t, x_0)\right)$ the Jacobian matrix of system (3.6), where $f(t, x_0)$ is a solution of system (3.6).

Consider two close points x_0 and $(x_0 + u_0)$ in the phase space U, where u_0 is a small disturbance of the point x_0. Then the evolution of vector $u(t) = f(t, x_0 + u_0) - f(t, x_0)$ can be studied Parker and Chua [1989] by the following linearized system

$$\dot{u} = A(t)u. \tag{3.7}$$

The solution of equation (3.7) can be represented as $u(t) = \Phi(t)u_0$, where $\Phi(t) = T_{x_0} f(t, x_0)$ is a fundamental matrix of system (3.7). The exponential rate of divergence (or convergence) of nearby trajectories is given by formula

$$\lambda(x_0, u_0) := \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \frac{\|u(t)\|}{\|u_0\|} = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|\Phi(t)u_0\|. \tag{3.8}$$

This value is called Lyapunov exponent of order 1 (or, simply, Lyapunov exponent).

It can be considered a generalization of Lyapunov exponent of order 1 to the case of order p, $1 \leq p \leq n$. Let E_0^p be the p-dimensional subspace of tangent space E_0 and U_0 be the open parallelepiped generated by p linearly independent vectors $e_1, \ldots, e_p$ of E_0^p. Then Lyapunov exponent of order p is defined Benettin et al. [1980a] as

$$\lambda^p(x_0, E_0^p) := \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \mathrm{Vol}^p(T_{x_0} f(t, U_0)) = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \mathrm{Vol}^p[\Phi(t)e_1, \ldots, \Phi(t)e_p], \tag{3.9}$$

where Vol^p means p-dimensional volume induced in tangent space by scalar product.

If in (3.8), (3.9) $\overline{\lim\limits_{t\to\infty}}$ can be replaced by $\lim\limits_{t\to\infty}$, then it is said that exact Lyapunov exponent exists.

It is known Lyapunov [1892]; Oseledec [1968]; Benettin et al. [1980a] that for regular linear systems there exist exact Lyapunov exponents[3] of order p, $1 \le p \le n$ and in the tangent space E_0 at the point x_0 it can be chosen p linearly independent vectors $e_1, \dots, e_p$ such that

$$\lambda^p(x_0, E_0^p) = \lambda_1(x_0) + \dots + \lambda_p(x_0), \tag{3.10}$$

where $\lambda_i(x_0) := \lambda(x_0, e_i)$, $i = 1 \dots p$, and $\lambda_1(x_0) \ge \dots \ge \lambda_p(x_0)$. That is, each Lyapunov exponent of order p is equal to the sum of p largest Lyapunov exponents of order 1.

In order to calculate all tangent vectors one can solve system (3.6) together with the matrix-valued variational equation Parker and Chua [1989]

$$\dot{\Phi}_t(x_0) = A(t)\,\Phi_t(x_0), \quad \Phi_0(x_0) = I, \tag{3.11}$$

where $\Phi_t(x_0) = T_{x_0} f(t, x_0)$ and I is identity matrix.

In this case one can go directly to the description of computation procedure. Choose the initial point x_0 and $(n \times n)$ matrix of orthonormal vectors $Q_0 = [q_1^0, \dots, q_n^0]$. During the k-th iteration, original system (3.6) is integrated together with variational equation (3.11) with the initial data $\{x_{k-1}, Q_{k-1}\}$ over the chosen small time interval h for obtaining $x_k = f(hk, x_0)$ and

$$U_k = [u_1^k, \dots, u_n^k] = \Phi_{hk}(x_0).$$

Then the matrix U_k is QR decomposed, i.e. $U_k = Q_k R_k$, where Q_k is orthogonal matrix and R_k is upper triangular matrix. The p-dimensional volume, defined in (3.9), increases by the multiplier $R_k(1,1) \cdots R_k(p,p)$ since $\mathrm{V}^p\{u_1^k, \dots, u_p^k\} = R_k(1,1) \cdots R_k(p,p)$, where $R_k(i,i)$ is a norm of the vector u_i^k, $i = 1 \dots p$. The matrix Q_k is taken as the initial datum for variational equation at the following iteration.

So, formula (3.9) can be expressed as

$$\lambda^p(x_0, U_0) = \lim_{k \to \infty} \frac{1}{kh} \sum_{i=1}^{k} \ln(R_i(1,1) \cdots R_i(p,p)), \quad 1 \le p \le n.$$

One repeats this iteration procedure K times. Subtracting λ^{p-1} from λ^p and using formula (3.10), one obtains approximate values of p-th Lyapunov exponent of order 1 for the chosen trajectory. By formula (2.4) a local Lyapunov dimension can also be computed.

3.2 Discussion and results

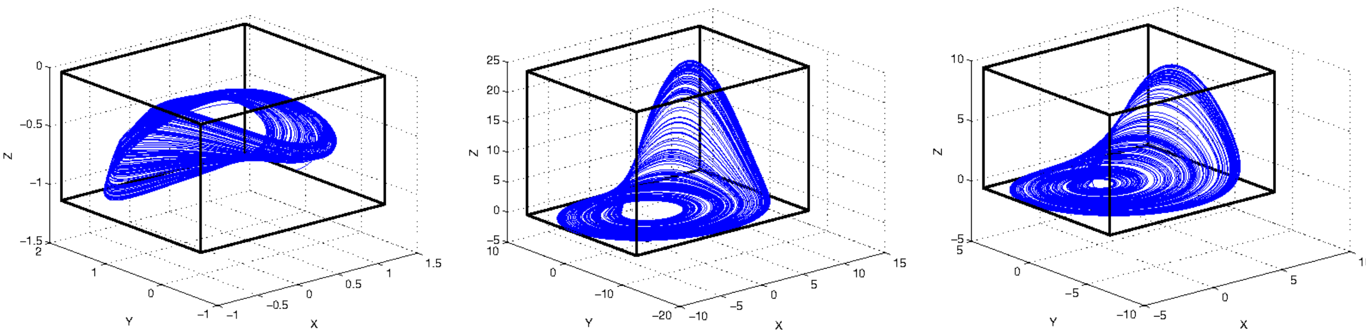
The algorithm, described in the previous section, is used in the process of justification of Leonov's conjecture. The entire computational procedure is implemented in MATLAB. For the orthogonalization of fundamental matrix it is used MATLAB library function *qr*, which implements a factorization procedure by using the Householder transformation since a classical Gram-Schmidt algorithm is numerically unstable and its modified version requires more execution time.

For nonlinear systems (2.1) there are no exact formulas, describing the solutions of these systems in general form. In this case it is considered approximated solutions, obtained by numerical integration of this systems, which is based on various finite-difference and more complex methods Yan and Ruan [2000]; Al-Sawalha and Noorani [2009]. For Rössler system (1.2) the problem of analysis of its analytical and numerical solutions is considered in Letellier et al. [2004].

In this paper for the integration of systems (2.1) it is used MATLAB realization (solver ode45) of Runge-Kutta finite-difference schemes of order 4-5 with an adaptive step. The absolute and relative tolerance are

[3] The opposite is not true: in the general case the existence of exact Lyapunov exponents does not imply regularity of the system Leonov and Kuznetsov [2007].

chosen equal to 10^{-8} since smaller values strongly influence a time of evaluation procedure. The parameter of procedure h, which determines integration time at each iteration, is chosen sufficiently small for the columns of fundamental matrix to be remained linearly independent. The parameter K – a number of iterations – must be sufficiently large in order that the trajectory, with the initial point in the neighborhood of attractor, covered this attractor. For the chosen parameters it was made the following: the number of iterations was increased by 2 times and a step was decreased by 2 times, in which case the result was qualitatively the same.



(a) Attractor localization for (1.1) (b) Attractor localization for (1.2) (c) Attractor localization for (1.3)

Figure 1: Localization of attractors of systems (2.1)

Since for Rössler systems (2.1) there are no analytical estimations of localization of their attractors, for estimation it is used computer experiments Barrio et al. [2009, 2011]. For the considered systems (2.1) their attractors are numerically localized in cubes (Fig. 1) by standard computational procedure[4]. On each cube it is chosen a grid with a certain step and at each grid point it is started the algorithm of computation of local Lyapunov dimension[5]. The obtained values are compared with a local Lyapunov dimension at stationary point. Then it is considered the grid points having the values of Lyapunov dimension, which are most close to a value at stationary point. Around each of these grid points it is considered a grid with a smaller step and at the points of this grid it is computed local Lyapunov dimensions. These values are also compared with a value at stationary point.

Table 1: *The results of justification for the following parameters:* $h = 1$, $K = 200$, $abs\_tol = rel\_tol = 10^{-8}$.

Rössler system	Cube	Grid step	$\max_{x \in grid} \dim_L x$	$\dim_L x_0$
(1.1)	$[-1; 1,3] \times [-0,7; 1,8] \times [-1,05; -0,03]$	0,1	2,4205	2,6042
(1.2)	$[-9; 12] \times [-11; 8] \times [-0,1; 23,9]$	0,5	2,0296	2,0341
(1.3)	$[-5; 7] \times [-7; 4] \times [-0,2; 9,8]$	0,5	2,0340	2,0620

[4] From a computational point of view, in nonlinear dynamical systems, attractors can be regarded as *self-excited* and *hidden attractors* Leonov et al. [2011c]; Bragin et al. [2011]; Leonov et al. [2012]; Leonov G. A. [2013]. Self-excited attractors can be localized numerically by *standard computational procedure, in which after a transient process a trajectory, started from a point of unstable manifold in a neighborhood of equilibrium, reaches a state of oscillation and therefore it can easily be identified.* In contrast, for a *hidden attractor, its basin of attraction does not intersect with small neighborhoods of equilibria.* While many classical attractors are self-exited attractors and therefore can be obtained numerically by standard computational procedure, for localization of hidden attractors it is necessary to develop special procedures since there are no similar transient processes leading to such attractors.

[5] Since numerical localization of attractors is considered and there is no effective way to prove ergodicity rigourously, one has to consider a mesh of initial conditions for investigation of Lyapunov exponents.

4 Conclusion

In this work Leonov's conjecture on Lyapunov dimension of various Rössler systems with standard parameters is verified numerically. While the data, given in Table (1), numerically confirm Leonov's conjecture, analytical proof of Leonov's conjecture is still an open problem.

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Appendix: Computation of Lyapunov exponents and Lyapunov dimension in MATLAB

Here it is given the main parts of program, written in MATLAB, which implements the described above algorithm for the computation of Lyapunov dimension of three-dimensional dynamical system (f.e. it is considered Rössler system (1.1)).

Listing 1: Computation of Lyapunov exponents

```
function [t, lces, trajectory] = lyapunov_exp(ode, x_start, t_start, ...
                                              t_step, k_iter, rel_tol, abs_tol)

% For given dynamical system, represented by system of differential equations
% combined with variational equation this function returns array of
% LCEs for the point x_start.
%
% Parameters:
%   ode - combined system (system of ode + var. eq.);
%   x_start -  initial point;
%   t_start - initial time value;
%   t_step - time-step in Gramm-Shmidt reorthogonalization procedure;
%   k_iter - number of iterations of Gramm-Shmidt reorthogonalization procedure;
%   rel_tol - relative error in Runge-Kutta 45 method;
%   abs_tol - absolute error in Runge-Kutta 45 method;

% n1 - size of the system of odes :
[~,n1] = size(x_start);

% n2 - size of combined system :
n2 = n1*(n1+1);

% Memory allocation (to increase the speed)

% y - variable of combined system :
y = zeros(n2,1);

% norms - array of norms of vectors in Jacobi matrix :
norms = zeros(1,n1);

% log_sum - array of sums of logarithms of norms :
log_sum = zeros(1,n1);

% l_exp - array of lyapunov exponents (in current moment) :
lexp = zeros(1,n1);

% Initializing y :
y(1:n1) = x_start(:);

for i = 1:n1
    y((n1+1)*i) = 1.0;
end

% Initializing t_curr :
t_curr = t_start;

% Preallocations for output values :
t = zeros(k_iter,1);
lces = zeros(k_iter,3);

% Set options for MATLAB solver :
options = odeset('RelTol', rel_tol, 'AbsTol', abs_tol);

tr_len = 1;
% Main loop:
for i = 1 : k_iter

    % Solving combined system :
    sol = ode45(ode, [t_curr t_curr+t_step], y, options);
    % i_last - the last moment :
    i_last = numel(sol.x);

    % Getting Jacobi matrix in the moment T PhiT
    % from vector Y :
    Y = transpose(sol.y);
    PhiT = reshape( Y(i_last, n1+1 : n2 ), n1, n1);

    % QR factorization of PhiT :
    [V, R] = qr(PhiT);

    for j = 1 : n1
        if R(j,j) < 0
            R(j,j) = (-1) * R(j,j);
            V(:,j) = (-1) * V(:,j);
        end
    end

    % Updating  y and t_curr :
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```
    t_curr = t_curr + t_step;
    y( 1 : n1 ) = Y( i_last, 1:n1 );
    y( n1+1 : n2 ) = reshape(V, 1, []);

    % Computing lyapunov exponents (in moment t_curr) :
    for k = 1 : n1
        norms(k) = R(k,k);
        log_sum(k) = log_sum(k) + log( norms(k) );
        lexp(k) = log_sum(k) / (t_curr-t_start);
    end

    % Saving computations in corresponding vectors :
    t(i) = t_curr;
    lces(i, :) = lexp;

    for j = 1 : i_last
        trajectory(tr_len, :) = [sol.x(j) sol.y(1:n1, j)'];
        tr_len = tr_len + 1;
    end
end
end
```

Listing 2: Computation of Lyapunov dimension

```
function ld = lyapunov_dim(lces)
% For the given array of lyapunov characteristic
% exponents of some point this function
% compute so called lyapunov dimention of
% this point.

% ld - lyupunov dimention :
ld = 0;

% n - number of LCEs :
[~,n] = size(lces);

% lambda - sorted array of LCEs :
lambda = sort(lces, 'descend');

% Main loop :
le_sum = lambda(1);
if ( lambda(1) > 0 )
     for i = 1 : n-1
        if lambda(i+1) ~= 0
           ld = i + le_sum / abs( lambda(i+1) );
           le_sum = le_sum + lambda(i+1);
           if le_sum < 0
              break;
           end
        end
    end
end
end
```

Listing 3: Rössler system (1.1)

```
function OUT = rossler_syst_1(t, X)

% Parameters:
global a b

% Output vector, that representing combined system:
OUT = zeros(12,1);

% Rosler equation:
OUT(1) = - X(2) - X(3);
OUT(2) = X(1);
OUT(3) = -b*X(3) + a*(X(2) - X(2)*X(2));

% Variational equation:
OUT(4:12) = [0 -1 -1; 1 0 0; 0 a*(1-2*X(2)) -b] ...
    * [X(4) X(7) X(10); X(5) X(8) X(11); X(6) X(9) X(12)];
```

Listing 4: Numerical procedure for Rössler system (1.1)

```
function run_rossler1

% Computes local lyapunov dimention in fixed point
% and in the points on the grid for the 1st Rossler
% attractor and compares thems.

% Parameters :
```

```
global a b

% Values of parameters :
a = 0.386; b = 0.2;

% T - time-step in iterative procedure :
T = 1.0;

% K - number of iterations of iterative procedure :
K = 200;

% Relative and absolute errors for Runge-Kutta 45 method :
rel_tol = 1e-8;
abs_tol = 1e-8;

% Epsilon -- is step on the grid :
eps = 1e-1;

% Fixed point :
x0 = [0 0 0];

% Attractor is located in cube :
x_begin = -1;    x_end = 1.3;  % x \in [-1; 1.3];
y_begin = -0.7;  y_end = 1.8;  % y \in [-0.7; 1.8];
z_begin = -1.1;  z_end = 0;    % z \in [-1.1; 0];

x_iterations = (x_end - x_begin) / eps + 1;
y_iterations = (y_end - y_begin) / eps + 1;
z_iterations = (z_end - z_begin) / eps + 1;

% Infinity factor: if trajectory leaves cube with side 'infinity_factor',
% then we conclude, that trajectory will leave basin of attraction :
infinity_factor = 10;

% Result array :
grid_results = zeros(x_iterations*y_iterations*z_iterations, 7);
i_res = 1;

% Looping the attractor grid :
for i = 1 : x_iterations
    for j = 1 : y_iterations
        for k = 1 : z_iterations

            % Main logic :
            curr_point = [x_begin+(i-1)*eps y_begin+(j-1)*eps z_begin+(k-1)*eps];
            [~, lces, trajectory] = lyapunov_exp(@rossler_syst_1, curr_point, 0, ...
                                                         T, K, rel_tol, abs_tol);
            len = size(trajectory, 1);

            if (abs(trajectory(len, 2)) < infinity_factor ...
                        && abs(trajectory(len, 3)) < infinity_factor ...
                                   && abs(trajectory(len, 4)) < infinity_factor)

            % Saving results for current point :
            grid_results(i_res, :) = [curr_point lyapunov_dim(lces(end, : )) ...
                                                                  lces(end, : )];
            i_res = i_res + 1;
            end
        end
    end
end

% Computing (local) lyapunov dimention for the fixed point :
[~, lces, ~] = lyapunov_exp(@rossler_syst_1, x0, 0, T, K, rel_tol, abs_tol);
LCEs = lces(end, : );


% Saving results in file :
fid = fopen('hypothesis_roessler_1.txt');
fprintf(fid, '%4s %4s %4s %10s %10s %10s %10s\r\n',...
                                'x', 'y', 'z', 'dim_L', 'lce1', 'lce3', 'lce3');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', grid_results);
fprintf(fid, '\r\nLyapunov dimension in fixed point:\r\n');
fprintf(fid, '%.2f, %.2f, %.2f, %.8f, %.8f, %.8f, %.8f\r\n', ...
                                                [x0 lyapunov_dim(LCEs) LCEs]);
fclose(fid);

end
```