IEEE Transactions on Circuits and Systems Part II: Express Briefs, 59(10), 2012, pp. 633-637 (doi:10.1109/TCSII.2012.2213362)

Analytical method for computation of phase-detector characteristic

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Draft [1] [2]

Abstract. Discovery of undesirable hidden oscillations, which cannot be found by simulation, in models of phase-locked loop (PLL) showed the importance of development and application of analytical methods for the analysis of such models. Approaches to a rigorous nonlinear analysis of analog PLL with multiplier phase detector (classical PLL) and linear filter are discussed. An effective analytical method for computation of multiplier/mixer phase-detector characteristics is proposed. For various waveforms of high-frequency signals, new classes of phase-detector characteristics are obtained, and dynamical model of PLL is constructed.

Keywords: nonlinear analysis of phase-locked loop (PLL), phase detector characteristics computation, simulation in phase-frequency space, hidden oscillation, hidden attractor

1 Introduction

Discovery of undesirable hidden oscillations[3] which cannot be found by simulation, in phase-locked loop (PLL) models showed the importance of development and application of analytical methods for the analysis of such models.

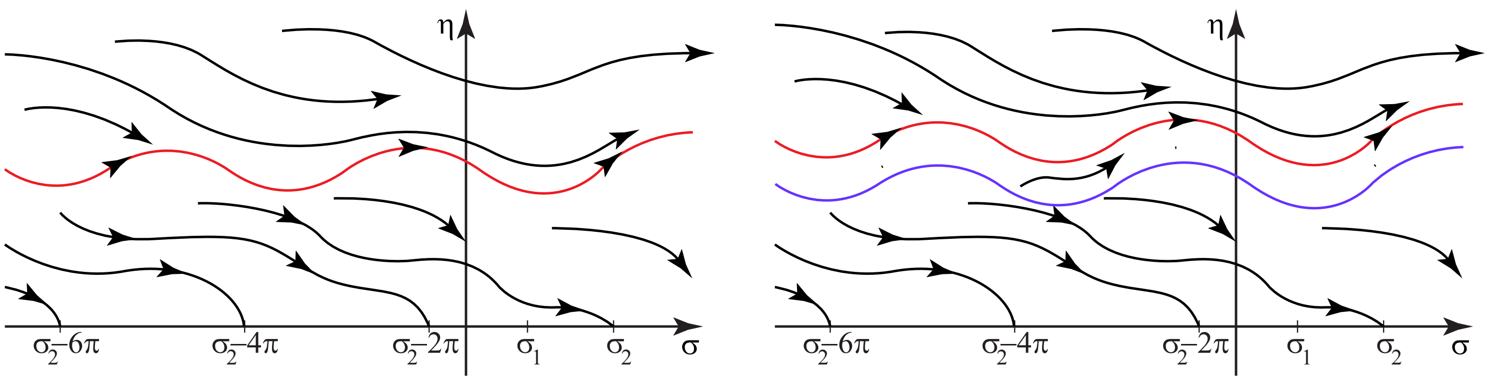

Figure 1: Bifurcation of hidden oscillation: stable and unstable periodic trajectory are bifurcated from the semistable periodic trajectories. If stable and unstable periodic solutions are very close one another, then from a computational point of view, all the trajectories tend to equilibria, but, in fact, there is a bounded domain of attraction only.

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[2]PDF slides http://www.math.spbu.ru/user/nk/PDF/Nonlinear-analysis-of-Phase-locked-loop-PLL.pdf

[3] From computation point of view, in nonlinear dynamical systems attractors can be regarded as *self-excited* and *hidden attractors*. *Self-excited attractors* can be localized numerically by *standard computational procedure* — after transient process a trajectory, started from a point of unstable manifold in a small neighborhood of unstable equilibrium, reaches an attractor and computes it. *Hidden attractors*, a basin of attraction of which does not contain neighborhoods of equilibria [2]. Computation of hidden attractors by standard computational procedure is impossible and requires application of special analytical-numerical procedures [1, 2, 3].

To carry out the nonlinear analysis of PLL, it is necessary to consider PLL models in signal and phase-frequency spaces [5, 6, 7, 8]. For constructing an adequate nonlinear mathematical model of PLL in phase-frequency space, it is necessary to find the characteristic of phase detector (PD) (PD is a nonlinear element used in PLL to match tunable signals). The PD inputs are high-frequency signals of reference and tunable oscillators, and the output contains a low-frequency error correction signal, corresponding to a phase difference of input signals. For the suppression of high-frequency component at PD output (if such component exists), a low-pass filter is applied. The characteristic of PD is a function defining a dependence of signal at the output of PD (in the phase-frequency space) on the phase difference of signals at the input of PD. PD characteristic depends on the realization of PD and waveforms of input signals.

The characteristics of classical PD-multiplier for typical sinusoidal signal waveforms are well known to engineers [5, 9, 10, 11, 12].

Furthermore, following [13], on the examples of PD in the form of multiplier, the general principles of computing phase detector characteristics for various types of signals, based on a rigorous mathematical analysis of high-frequency oscillations [14, 16], will be considered.

2 Description of Classical PLL in Signal Space

Consider classical PLL on the level of electronic realization (Fig. 2)

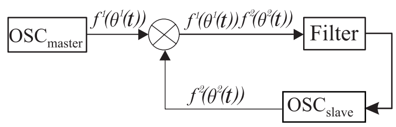


Figure 2: Block diagram of PLL on the level of electronic realization.

Here $\text{OSC}_{\text{master}}$ is a master oscillator, and $\text{OSC}_{\text{slave}}$ is a slave oscillator [tunable voltage-control oscillator (VCO)], which generates oscillations $f^p(t) = f^p(\theta^p(t)),\ p = 1, 2$ with $\theta^p(t)$ as phases, correspondingly.

The block $\bigotimes$ is a multiplier (used as PD) of oscillations $f^1(t)$ and $f^2(t)$, and the signal $f^1(\theta^1(t))f^2(\theta(t))$ is its output. The relation between the input $\xi(t)$ and the output $\sigma(t)$ of linear filter is as follows:

$$\sigma(t) = \alpha_0(t) + \int_0^t \gamma(t-\tau)\xi(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\gamma(t)$ is an impulse response function of filter and $\alpha_0(t)$ is an exponentially damped function depending on the initial data of filter at moment $t = 0$. By assumption, $\gamma(t)$ is a differentiable function with bounded derivative (this is true for the most considered filters [11]).

2.1 High-frequency property of signals

Suppose that the waveforms $f^{1,2}(\theta)$ are bounded 2π-periodic piecewise differentiable functions[4] (this is true for the most considered waveforms). Consider Fourier series representation of such functions

$$\begin{aligned} f^p(\theta) &= \sum_{i=1}^{\infty} \big(a_i^p \sin(i\theta) + b_i^p \cos(i\theta)\big),\ p = 1, 2,\\ a_i^p &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\sin(i\theta)d\theta, b_i^p = \frac{1}{\pi}\int_{-\pi}^{\pi} f^p(\theta)\cos(i\theta)d\theta. \end{aligned}$$

A high-frequency property of signals can be reformulated in the following way. By assumption, the phases $\theta^p(t)$ are smooth functions (this means that frequencies are changing continuously, which is corresponding to

[4]the functions with a finite number of jump discontinuity points differentiable on their continuity intervals

classical PLL analysis [11, 12]). Suppose also that there exists a sufficiently large number ω_{min} such that the following conditions are satisfied on the fixed time interval $[0, T]$:

$$\dot\theta^p(\tau) \geq \omega_{min} > 0, \quad p = 1, 2 \tag{2}$$

where T is independent of ω_{min} and $\dot\theta^p(t)$ denotes frequencies of signals. The frequencies difference is assumed to be uniformly bounded

$$\left|\dot\theta^1(\tau) - \dot\theta^2(\tau)\right| \leq \Delta\omega, \ \forall\tau \in [0, T]. \tag{3}$$

Requirements (2) and (3) are obviously satisfied for the tuning of two high-frequency oscillators with close frequencies [11, 12]. Let us introduce $\delta = \omega_{min}^{-\frac{1}{2}}$ Consider the relations

$$\begin{aligned} &|\dot\theta^p(\tau) - \dot\theta^p(t)| \leq \Delta\Omega, \ p = 1, 2, \\ &|t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T], \end{aligned} \tag{4}$$

where $\Delta\Omega$ is independent of δ and t. Conditions (2)–(4) mean that the functions $\dot\theta^p(\tau)$ are almost constant and the functions $f^p(\theta^p(\tau))$ are rapidly oscillating on small intervals $[t, t + \delta]$.

The boundedness of derivative of $\gamma(t)$ implies

$$|\gamma(\tau) - \gamma(t)| = O(\delta), \quad |t - \tau| \leq \delta, \quad \forall\tau, t \in [0, T]. \tag{5}$$

3 Phase-Detector Characteristic Computation

Consider two block diagrams shown in Fig. 3. Here, PD is a nonlinear block with characteristic $\varphi(\theta)$. The

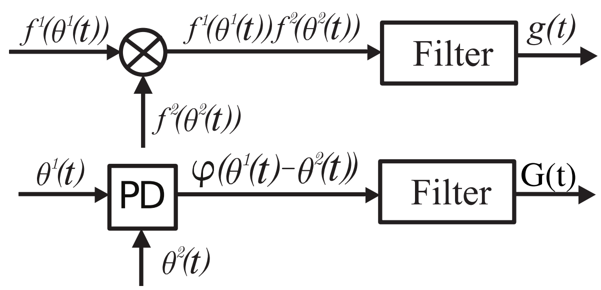


Figure 3: Phase detector and filter

phases $\theta^p(t)$ are PD block inputs, and the output is a function $\varphi(\theta^1(t) - \theta^2(t))$. The PD characteristic $\varphi(\theta)$ depends on waveforms of input signals.

The signal $f^1(\theta^1(t))f^2(\theta^2(t))$ and the function $\varphi(\theta^1(t) - \theta^2(t))$ are the inputs of the same filters with the same impulse response function $\gamma(t)$ and with the same initial state. The outputs of filters are functions $g(t)$ and $G(t)$, respectively. By (1), one can obtain $g(t)$ and $G(t)$

$$\begin{aligned} g(t) &= \alpha_0(t) + \int_0^t \gamma(t - \tau) f^1\big(\theta^1(\tau)\big) f^2\big(\theta^2(\tau)\big) d\tau, \\ G(t) &= \alpha_0(t) + \int_0^t \gamma(t - \tau) \varphi\big(\theta^1(\tau) - \theta^2(\tau)\big) d\tau. \end{aligned} \tag{6}$$

Then, using the approaches outlined in [13] and [17, 18, 26], the following result can be proved.

Theorem 1 *Let conditions* (2)–(5) *be satisfied and*

$$\varphi(\theta)=\frac{1}{2}\sum_{l=1}^{\infty}\Big((a_l^1 a_l^2 + b_l^1 b_l^2)\cos(l\theta) + (a_l^1 b_l^2 - b_l^1 a_l^2)\sin(l\theta)\Big). \tag{7}$$

Then the following relation:

$$|G(t) - g(t)| = O(\delta), \quad \forall t \in [0, T]$$

is valid.

Proof. Suppose that $t \in [0, T]$. Consider the difference

$$g(t) - G(t) = \int_0^t \gamma(t-s)\Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \\ - \varphi\big(\theta^1(s) - \theta^2(s)\big)\Big]ds. \tag{8}$$

...

it can be obtained that

$$g(t) - G(t) = \sum_{k=0}^{m}\gamma(t-k\delta)\int\limits_{[k\delta,(k+1)\delta]} \\ \Big[f^1\big(\theta^1(s)\big)f^2\big(\theta^2(s)\big) - \varphi\big(\theta^1(s) - \theta^2(s)\big)\Big]ds + O(\delta). \tag{9}$$

...

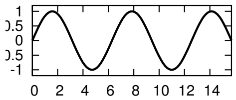
The proof of theorem is completed. ■

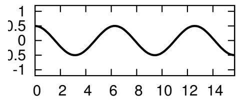
Roughly speaking, this theorem separates low-frequency error-correcting signal from parasite high-frequency oscillations. This result was known to engineers [11] for sinusoidal signals.

This theorem allows one to compute a phase detector characteristic for the following typical signals [11] shown in the table hereinafter. The waveforms $f^{1,2}(\theta)$ of input signals are shown in the left diagram and the corresponding PD characteristic $\varphi(\theta)$ is plotted in the right one.

3.1 Phase detector characteristics for equal signal waveforms

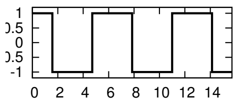
$f^{1,2}(\theta) = \sin(\theta)$, $\varphi(\theta) = \frac{1}{2}\cos(\theta)$

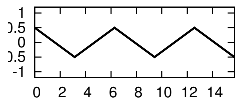




$f^{1,2}(\theta) = \operatorname{sign}\left(\sin(\theta)\right)$,

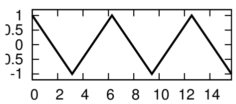
$$\varphi(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big)$$

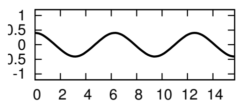




$$f^{1,2}(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$$

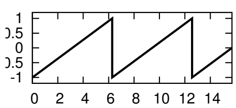
$$\varphi(\theta) = \frac{32}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\big((2n-1)\theta\big)$$

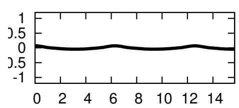




$$f^{1,2}(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\big(n\theta\big),$$

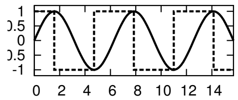
$$\varphi(\theta) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\big(n\theta\big)$$

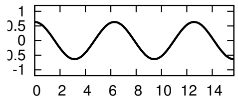




3.2 Phase detector characteristics for mixed signal waveforms

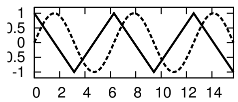
$f^1(\theta) = \sin(\theta),\ f^2(\theta) = \text{sign}\ \sin(\theta),\ \varphi(\theta) = \frac{2}{\pi}\cos(\theta)$

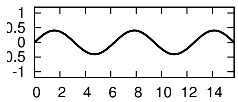




$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$

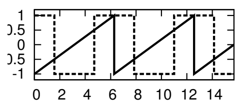
$f^2(\theta) = \sin(\theta),\ \varphi(\theta) = \frac{4}{\pi^2}\sin(\theta)$

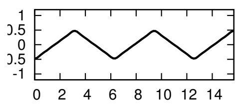




$f^1(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\big(n\theta\big),\ f^2(\theta) = \text{sign}\ \sin(\theta),$

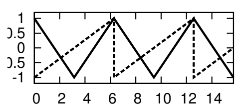
$\varphi(\theta) = -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\theta)}{(2n-1)^2}$

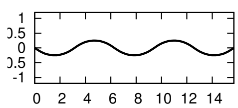




$f^1(\theta) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\big(n\theta\big),\ \varphi(\theta) = -\frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$

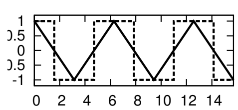
$f^2(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$

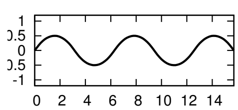




$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$

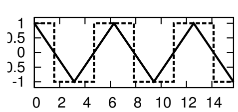
$f^2(\theta) = \text{sign}\ \sin(\theta),\ \varphi(\theta) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$

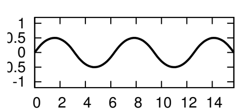




$f^1(\theta) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\big((2n-1)\theta\big),$

$f^2(\theta) = \text{sign}\ \sin(\theta),\ \varphi(\theta) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\theta)}{(2n-1)^3}$





4 Description of Classical PLL in Phase-Frequency Space

From the mathematical point of view, a linear filter can be described [11] by a system of linear differential equations

$$\dot{x} = Ax + b\xi(t), \ \sigma = c^* x, \tag{10}$$

a solution of which takes the form (1). Here, A is a constant matrix, $x(t)$ is a state vector of filter, b and c are constant vectors.

The model of tunable generator is usually assumed to be linear [11, 12]:

$$\dot{\theta}^2(t) = \omega^2_{free} + LG(t), \ t \in [0, T]. \tag{11}$$

where ω^2_{free} is a free-running frequency of tunable generator and L is an oscillator gain. Here it is also possible to use nonlinear models of VCO; see, e.g., [20] and [21]).

Suppose that the frequency of master generator is constant $\dot{\theta}^1(t) \equiv \omega^1$. Equation of tunable generator (11) and equation of filter (10), yield

$$\dot{x} = Ax + bf^1(\theta^1(t))f^2(\theta^2(t)), \ \dot{\theta}^2 = \omega^2_{free} + Lc^* x. \tag{12}$$

The system (12) is nonautonomous and rather difficult for investigation [6]. Here, Theorem 1 allows one to study more simple autonomous system of differential equations [in place of the nonautonomous (12)]

$$\begin{aligned} &\dot{x} = Ax + b\varphi(\Delta\theta), \ \Delta\dot{\theta} = \omega^2_{free} - \omega^1 + Lc^* x, \\ &\Delta\theta = \theta^2 - \theta^1. \end{aligned} \tag{13}$$

Well-known averaging method [22, 23, 24] allows one to show that solutions of (12) and (13) are close under some assumptions. Thus, by Theorem 1, the block-scheme of PLL in signal space (Fig. 2) can be asymptotically changed [for high-frequency generators, see conditions (2)–(4)] to the block-scheme on the level of phase-frequency relations (Fig. 4).

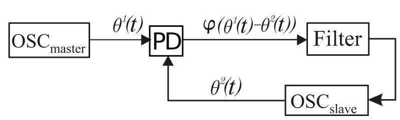


Figure 4: Block scheme of phase-locked loop in phase-frequency space

In Fig. 4, PD has the corresponding characteristics. Thus, using asymptotic analysis of high-frequency oscillations, the characteristics of PD can be computed. Methods of nonlinear analysis for this model are well developed [6].

The simulation approach for PLL analysis and design, based on the obtained analytical results, is discussed in [25].

It should be noted that, instead of conditions (3) and (5) for simulations of real system, one have to consider the following conditions:

$$|\Delta\omega| << \omega_{min}, \ |\lambda_A| << \omega_{min},$$

where λ_A is the largest (in modulus) eigenvalue of the matrix A. Also, for correctness of transition from equation (8) to (9) one have to consider $T << \omega_{min}$. Theoretical results are justified by simulation of PLL model in phase-frequency space and signal space (Fig. 5). Unlike the filter output for the phase-frequency model, the output of the filter for signal space PLL model contains additional high-frequency oscillations. These high-frequency oscillations interfere with qualitative analysis and efficient simulation of PLL.

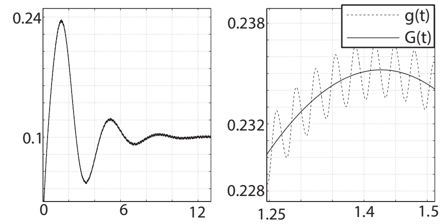


Figure 5: $\omega^2_{free} = 99$ Hz, $\omega^1 = 100$ Hz, $L = 10$, filter transfer functions $\frac{1}{s+1}$, and triangle waveforms

The passage to analysis of autonomous dynamical model of PLL (in place of the nonautonomous one) allows one to overcome the aforementioned difficulties, related with modeling PLL in time domain, which were noted in survey lecture of well-known American specialist D. Abramovitch at American Control Conference, 2008: One has to simultaneously observe *"very fast time scale of the input signals"* and *"slow time scale of signal's phase"*.

5 Conclusion

The approach, proposed in this brief, allows one (mathematically rigorously) to compute multiplier PD characteristics in the general case of signal waveforms and to proceed from analysis of classical PLL in time space to analysis and simulation in phase-frequency space. This allows one to effectively simulate classical PLL.

References

[1] G. A. Leonov, N. V. Kuznetsov, and V. I. Vagaitsev, "Hidden attractor in smooth hua's systems," *Physica D: Nonlinear Phenomena*, vol. 241, no. 18, pp. 1482–1486, 2012. doi:10.1016/j.physd.2012.05.016

[2] Leonov, G.A., Kuznetsov, N.V. and Vagaytsev V.I. (2011), Localization of hidden Chua's attractors, *Physics Letters A*, **375**(23), 2230–2233 (DOI: 10.1016/j.physleta.2011.04.037).

[3] Bragin, V.O., Kuznetsov, N.V., Leonov, G.A. and Vagaitsev, V.I. (2011), Algorithms for Finding Hidden Oscillations in Nonlinear, Systems. The Aizerman and Kalman Conjectures and Chua's Circuits, *Journal of Computer and Systems Sciences International*, **50**(4), 511–544 (DOI: 10.1134/S106423071104006X).

[4] N. A. Gubar, "Investigation of a piecewise linear dynamical system with three parameters," *J. Appl. Math. Mech*, vol. 25, no. 6, pp. 1519–1535, 1961.

[5] A. Viterbi, *Principles of coherent communications.* New York: McGraw-Hill, 1966.

[6] J. Kudrewicz and S. Wasowicz, *Equations of phase-locked loop. Dynamics on Circle, Torus and Cylinder*, ser. A. World Scientific, 2007, vol. 59.

[7] T. J. Yamaguchi, M. Soma, M. Ishida, T. Watanabe, and T. Ohmi, "Extraction of instantaneous and rms sinusoidal jitter using an analytic signal method," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 50, no. 6, pp. 288–298, 2003.

[8] G. Manganaro, S. U. Kwak, S. Cho, and A. Pulincherry, "A behavioral modeling approach to the design of a low jitter clock source," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 50, no. 11, pp. 804–814, 2003.

[9] D. Abramovitch, "Phase-locked loops: A control centric tutorial," in *Proc. of the American Control Conference*, vol. 1, 2002, pp. 1–15.

[10] W. F. Egan, *Frequency Synthesis by Phase Lock.* John Wiley&Sons, 2000.

[11] R. E. Best, *Phase-Lock Loops: Design, Simulation and Application.* Mcgraw-Hill, 2003.

[12] V. F. Kroupa, *Phase Lock Loops and Frequency Synthesis.* John Wiley & Sons, 2003.

[13] G. A. Leonov, "Computation of phase detector characteristics in phase-locked loops for clock synchronization," *Doklady Mathematics*, vol. 78, no. 1, pp. 643–645, 2008.

[14] G. A. Leonov and S. M. Seledzhi, "Stability and bifurcations of phase-locked loops for digital signal processors," *International journal of bifurcation and chaos*, vol. 15, no. 4, pp. 1347–1360, 2005.

[15] N. V. Kuznetsov, G. A. Leonov, and S. S. Seledzhi, "Phase locked loops design and analysis," in *ICINCO 2008 - 5th International Conference on Informatics in Control, Automation and Robotics, Proceedings*, vol. SPSMC. INSTICC Press, 2008, pp. 114–118. doi:10.5220/0001485401140118

[16] N. V. Kuznetsov, G. A. Leonov, S. M. Seledzhi, and P. Neittaanmäki, "Analysis and design of computer architecture circuits with controllable delay line," in *ICINCO 2009 - 6th International Conference on Informatics in Control, Automation and Robotics, Proceedings*, vol. 3 SPSMC. INSTICC Press, 2009, pp. 221–224. doi:10.5220/0002205002210224

[17] G. A. Leonov, N. V. Kuznetsov, M. V. Yuldashev, and R. V. Yuldashev, "Computation of phase detector characteristics in synchronization systems," *Doklady Mathematics*, vol. 84, no. 1, pp. 586–590, 2011. doi:10.1134/S1064562411040223

[18] N. V. Kuznetsov, G. A. Leonov, M. V. Yuldashev, and R. V. Yuldashev, "Analytical methods for computation of phase-detector characteristics and pll design," in *ISSCS 2011 - International Symposium on Signals, Circuits and Systems, Proceedings.* IEEE Press, 2011, pp. 7–10. doi:10.1109/ISSCS.2011.5978639

[19] N. V. Kuznetsov, P. Neittaanmäki, G. A. Leonov, S. M. Seledzhi, M. V. Yuldashev, and R. V. Yuldashev, "High-frequency analysis of phase-locked loop and phase detector characteristic computation," in *ICINCO 2011 - Proceedings of the 8th International Conference on Informatics in Control, Automation and Robotics*, vol. 1. IEEE Press, 2011, pp. 272–278. doi:10.5220/0003522502720278

[20] L. Xiaolue, W. Yayun, and R. Jaijeet, "Fast pll simulation using nonlinear vco macromodels for accurate prediction of jitter and cycle-slipping due to loop non-idealities and supply noise," in *Proceedings of the 2005 Asia and South Pacific Design Automation Conference.* New York: ACM, 2005, pp. 459 – 464.

[21] A. Demir, A. Mehrotra, and J. Roychowdhury, "Phase noise in oscillators: a unifying theory and numerical methods for characterization," *IEEE Trans. Circuits Syst. I*, vol. 47, pp. 655–674, 2000.

[22] N. M. Krylov and N. N. Bogolubov, *Introduction to Nonlinear Mechanics.* Princeton: Princeton University Press, 1947.

[23] F. C. Hoppensteadt and E. M. Izhikevich, *Weakly Connected Neural Networks.* Springer Verlag, 1997, pp. 248–292.

[24] I. G. Malkin, *Some Problems in Nonlinear Oscillation Theory.* Moscow: Gostexizdat, 1962.

[25] N. V. Kuznetsov, G. A. Leonov, M. V. Yuldashev, R. V. Yuldashev, and S. M. Seledzhi, “Method for determining the operating parameters of phase-locked oscillator frequency and device for its implementation,” Patent 2 010 149 471/08(071 509) (RU), Oct. 27, 2011.

[26] N. V. Kuznetsov, G. A. Leonov, S. M. Seledzhi, “Nonlinear analysis of the Costas loop and phase-locked loop with squarer,” in *Proceedings of Eleventh IASTED International Conference Signal and Image Processing*, vol. 654. Acta Press, 2009, pp. 1–7.