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Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system

Kuznetsov N.V., Kuznetsova O.A., Leonov G.A.

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Abstract. This paper is devoted to analytical and numerical investigation of limit cycles in two-dimensional polynomial quadratic systems. The appearance of modern computers permits one to use a numerical simulation of complicated nonlinear dynamical systems and to obtain new information on a structure of their trajectories. However the possibilities of naive approach, based on the construction of trajectories by numerical integration of the considered differential equations, turns out to be highly limited.

In the paper the effective analytical-numerical methods for investigation of limit cycles in two-dimensional polynomial quadratic system are discussed. Estimations of domains of parameters, corresponding to existence of different configurations of large limit cycles, are obtained and visualization of four large limit cycles in quadratic system is presented.

Keywords: Limit cycle, 16th Hilbert problem, quadratic system, visualization, Lyapunov value (Lyapunov quantities, coefficients, Poincare-Lyapunov constants, focus values)

1 Introduction

The study of limit cycles of two-dimensional dynamical systems was stimulated by purely mathematical problems (the center-and-focus problem, Hilbert's sixteenth problem, and isochronous centers problem) as well as many applied problems (the oscillations of electronic generators and electrical machines, the dynamics of populations). At the present time there are different methods for "construction" of limit cycles.

Today for the study of bifurcations of limit cycles it is used such methods as the investigation of Poincare mapping, the investigation of Poincare-Mel'nikov and Abel integrals, and the averaging methods (see, e.g., [1, 2, 3, 4] at al.) and others. But "small" parameters, usually used for numerical construction of limit cycles on the basis of these methods, often makes the task of numerical analysis of limit cycles difficult enough, especially in the case of nested limit cycles (hidden oscillations [5, 6]). The appearance of modern computers permits one to use numerical simulation of complicated nonlinear dynamical systems and to obtain new information on a structure of their trajectories. However the possibilities of "simple" approach, based on the construction of trajectories by numerical integration of the considered differential equations, turned out to be highly limited. This is shown, for example, in the task posed by academician A.N. Kolmogorov which is described by V.I. Arnol'd in [7]: *To estimate the number of limit cycles of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech & Math Faculty of Moscow State University as a mathematical practice. Each student had to find the number of limit cycles of a field. The result of this experiment was absolutely unexpected: not a single field had a limit cycle!* This experiment shows the necessity of developing special analytical-numerical methods for the investigation and numerical visualization of limit cycles.

In the present work the use of the method of disturbance of Lyapunov quantities together with the method of asymptotical integration allows one to obtain the conditions of existence of four limit cycles in quadratic

[1] PDF slides http://www.math.spbu.ru/user/nk/PDF/Limit-cycles-Focus-values-Lyapunov-quantity-16th-Hilbert.pdf

systems: three large limit cycles in the case of a weak focus of first order, two large limit cycles in the case of a weak focus of second order, one large limit cycle in the case of a weak focus of third order. The conditions obtained have very simple form and generalize the widely known Shi theorem. The development of analytical and numerical methods permits us to find first an example of quadratic system, for which four large limit cycles can be visualized.

2 Criteria of existence of limit cycles

Consider two-dimensional quadratic system

$$\begin{aligned} \frac{dx}{dt} &= a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y, \\ \frac{dy}{dt} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y, \end{aligned} \tag{2.1}$$

and its reduced form [8]

$$\begin{aligned} \frac{dx}{dt} &= x^2 + xy + y, \\ \frac{dy}{dt} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y. \end{aligned} \tag{2.2}$$

Then for system (2.2) with a help of the method of small disturbance of Lyapunov quantities and the method of asymptotical integration [8], the criterion of existence of 4 limit cycles (3 “small” and 1 “large” cycles or 2 “small” and 2 “large” cycles) can be stated [9]

Theorem 1 *System (2.2) has 4 limit cycles if the conditions*

$$c_2 \in (1/3, 1), \quad b_2 \in (1, 3), \quad 4a_2(c_2 - 1) > (b_2 - 1)^2, \quad b_2c_2 > 1,$$

$$\beta_2 \in (0, \varepsilon), \quad \alpha_2 \in (\frac{a_2(2 + b_2)}{b_2c_2 - 1}, \frac{a_2(2 + b_2)}{b_2c_2 - 1} + \delta), \quad 1 \gg \delta \gg \varepsilon \geq 0$$

are satisfied.

In Fig. 1 is shown the domain described by Shi (shaded) and the domain (grey) described by conditions of Theorem 1 with $b_2 = 3, \alpha_2 = \frac{a_2(2+b_2)}{b_2c_2-1}, \beta_2 = 0$ (the case of 1 “large” cycles and weak focus of third order, which allows to obtain additional three “small” limit cycles by small disturbance of parameters). So, domain obtained here involves entirely Shi domain [10], and corresponds to the domain obtained analytically for the same case in the work [11].

Note, that obtained here analytically domain can be extended [12] numerically.

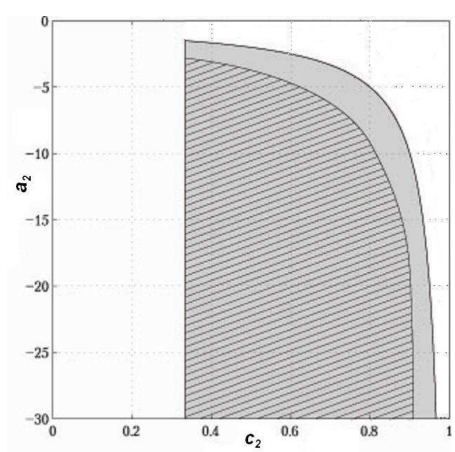

Figure 1: Shi domain and its expansion

Note that projection of 3-dimensional domain corresponding to the existence 2 “large” limit cycles with weak focus of second order ($\alpha_2 = \frac{a_2(2+b_2)}{b_2c_2-1}, \beta_2 = 0$) is shown in Fig. 2a. In Fig. 2b are shown the trajectories of quadratic system with coefficients from the above domain. In the domain of closeness of trajectories there are presented one stable (on the right) and one unstable (on the left) “large” limit cycles (2 “small” limit cycles around zero point can also be obtained by small disturbances of parameters of system).

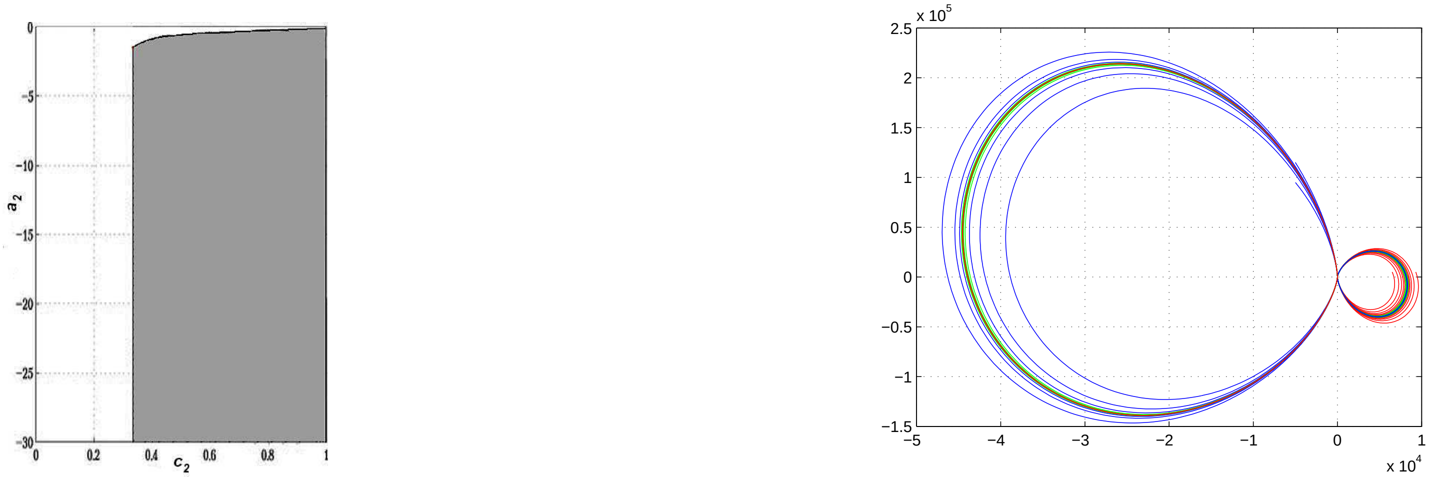

Figure 2: a. Projection of 3-dimensional domain (2 large cycle) b. Two “large“ cycles

In the work [13] it was formulated the following criterion of the existence of 1 ”small” and 3 ”large” limit cycles:

Theorem 2 *System (2.2) has 3 ”large” limit cycles if following conditions are satisfied*
$\alpha_2 = -\varepsilon^{-1}, \quad \beta_2 = 0, \quad c_2 \in (\frac{1}{3}, \frac{1}{2}), \quad b_2 > a_2 + c_2, \quad 2c_2 \leq b_2 + 1,$
$4a_2(c_2 - 1) > (b_2 - 1)^2, \quad b_2c_2 < 1,$ *where* $1 \gg \varepsilon \geq 0$.

Besides, 1 ”small” limit cycle can be obtained around the origin of coordinates by a disturbance of β_2.

Note that the condition $b_2c_2 < 1$, corresponding to that the inequality $L_1 < 0$ is satisfied, is not required.

Recall [14, 15] that $L_1(0) = \frac{-\pi}{4(\alpha_2)^{\frac{5}{2}}}(\alpha_2(b_2c_2 - 1) - a_2(b_2 + 2))$. Here the condition $L_1 < 0$ is satisfied for either $b_2c_2 < 1, \quad \alpha_2 < 0$, either $b_2c_2 > 1, \quad \alpha_2 < 0, \quad \alpha_2 > \frac{a_2(2+b_2)}{b_2c_2-1}$.

Note also that the condition $b_2 > a_2 + c_2$ is inessential requirement since it results from other conditions.

Further in the process of numerical experiments it was considered a problem on a possible enlargement of the domain of coefficients ($b_2, \ c_2$), which correspond to the existence of 3 ”large” and 1 ”small” limit cycles. According to these experiments, the conditions $2c_2 \leq b_2 + 1$ and $c_2 > \frac{1}{3}$ are necessary. While the condition $c_2 < \frac{1}{2}$ is not necessary. In this work it is experimentally obtained that for $c_2 < 1$ and $b_2 < 3$ there exist three ”large” (and 1 ”small”) limit cycles.

Thus, new conditions of the existence of 4 limit cycles (3 ”large” and 1 ”small”), obtained by means of numerical-analytical approach, can be formulated in the following way:

System (2.2) has 3 ”large” limit cycles if the following conditions are satisfied:
$\alpha_2 < 0$ *and* $|\alpha_2|$ *is sufficiently large,* $\beta_2 = 0, \quad c_2 \in (\frac{1}{3}, 1), c_2 \neq \frac{1}{2}, \quad b_2 < 3, \quad 2c_2 \leq b_2 + 1, \ 4a_2(c_2 - 1) > (b_2 - 1)^2$, *and either* $b_2c_2 < 1$, *or* $b_2c_2 > 1$ *and* $\frac{a_2(2+b_2)}{b_2c_2-1} < \alpha_2$.

Besides, 1 ”small” limit cycle can be obtained around the origin of coordinates in the case of disturbance of parameter β_2.

The projection of the above described domain of coefficients is shown in Fig. 3.

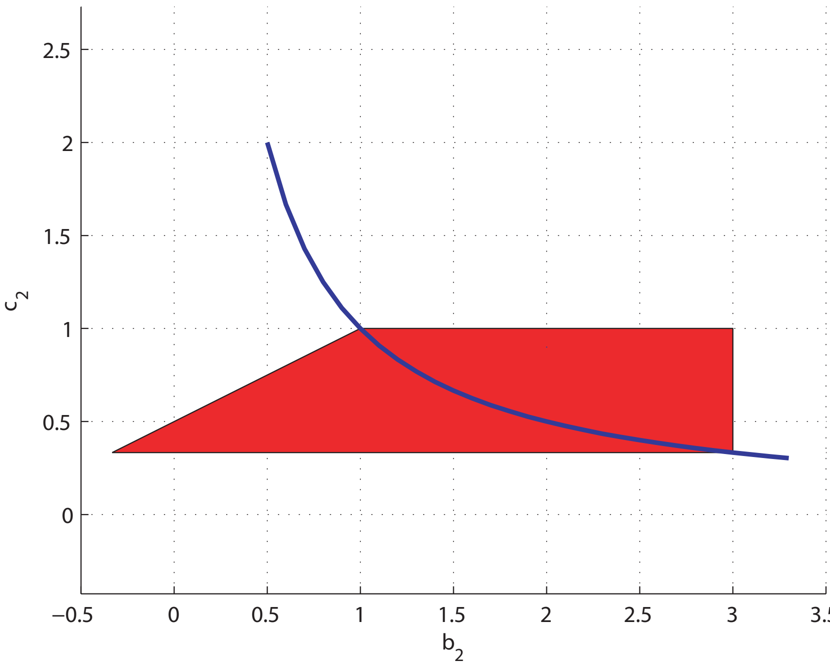

Figure 3: Projection of 3-dimensional domain (3+1 cycles)

Here appropriate disturbance of β_2 allows one to obtain the third "large" limit cycle around zero point in place of the "small" one. In Fig. 4 for coefficients $b_2 = 2.2, \quad c_2 = 0.7$ the coefficients $a_2 = -10, \quad \alpha_2 = -72.7778, \quad \beta_2 = 0.0015$ are selected. Localization of three "large" limit cycles around zero point and 1 "large" limit cycle to the left of straight line $x = -1$ can be observed.

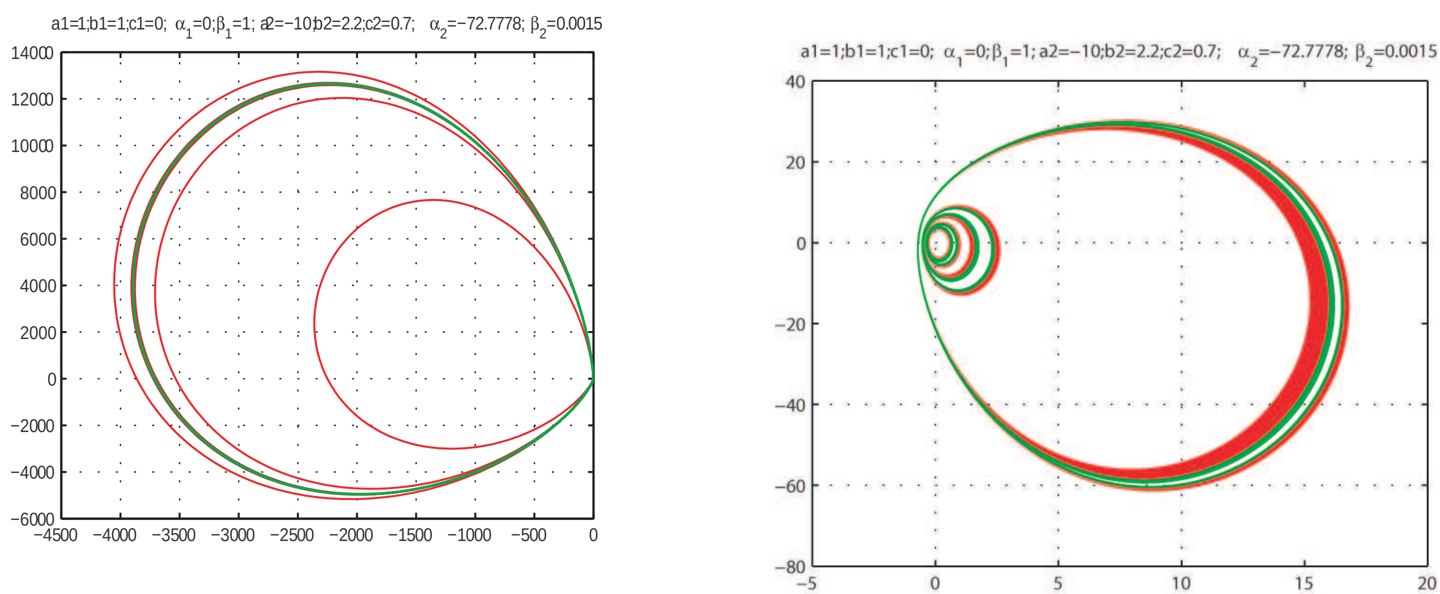

Figure 4: Visualization of four limit cycles in two-dimensional polynomial quadratic system

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Appendix: visualization of four limit cycles in Matlab

```
function f=fcyclesPlot(a1,b1,c1,al1,bt1,a2,b2,c2,al2,bt2, ->
    x0_1,x0_2,x0_3,x0_4,y0_1,y0_2,acc,len1,len2,len3,len4,->
    len5,len6,len7,len8)

  clear all; syms x
  global a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2
           %
  title_coefQs =['a1=',num2str(a1),';b1=',num2str(b1),->
  ';c1=',num2str(c1),';\alpha_1=',num2str(al1),->
  ';\beta_1=',num2str(bt1),';a2=',num2str(a2),';->
  b2=' num2str(b2),';c2= ',num2str(c2),';\alpha_2=',->
  num2str(al2), ';\beta_2=',num2str(bt2)];
  RelTol = acc; AbsTol = acc; InitialStep = acc; %
  options = odeset('RelTol',RelTol,'AbsTol',AbsTol, ->
  'InitialStep',InitialStep, 'NormControl','on');
       %                 ,                 1               3-
  x0=x0_1;y0=0;[T,XY]=ode45(@fQsys,[0 len1],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;
  x0=x0_2;y0=0;[T,XY]=ode45(@fQsys,[0 len2],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;

  x0=x0_2;y0=0;[T,XY]=ode45(@fQsys,[0 len3],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;
  x0=x0_3;y0=0;[T,XY]=ode45(@fQsys,[0 len4],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green');
       hold on; grid on;

  x0=x0_3;y0=0;[T,XY]=ode45(@fQsys,[0 len5],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;
  x0=x0_4;y0=0;[T,XY]=ode45(@fQsys,[0 len6],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;

  x0=-1000; y0=y0_1;[T,XY]=ode45(@fQsys,[0 len7],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;
  x0=-1000; y0=y0_2;[T,XY]=ode45(@fQsys,[0 len8],[x0 y0],options);
       fPlotTrajectory(XY(:,1),XY(:,2),'red','blue','green' );
       hold on; grid on;

  title({title_coefQs})
```

Additional functions

```
function f = fPlotTrajectory(X,Y,Color1,Color2,Color3)
          %
  lenTr = length(X); lenTr3 = round(lenTr/9);
  lenColor1 = round(4*lenTr3); lenColor2 =  round(7*lenTr3);
          %
  plot(X(1 :lenColor1),Y(1 :lenColor1),Color1); hold on;
  plot(X(lenColor1:lenColor2),Y(lenColor1:lenColor2),Color2);
  hold on;
  plot(X(lenColor2:length(X)),Y(lenColor2:length(Y)),Color3);
```

```
function dz = fQsys(t,z) %
  global a1 b1 c1 al1 bt1 a2 b2 c2 al2 bt2
  dz = zeros(2,1); % z = ( z(1),z(2) ) = (x,y)

  dz(1) = (a1*z(1)^2+b1*z(1)*z(2)+c1*z(2)^2+al1*z(1)+bt1*z(2));
  dz(2) =(a2*z(1)^2+b2*z(1)*z(2)+c2*z(2)^2+al2*z(1)+bt2*z(2));
```

For coefficients
$a1 = 1;\ b1 = 1;\ c1 = 0;\ al1 = 0;\ bt1 = 1;\ a2 = -10;\ b2 = 2.7;\ c2 = 0.4;$
$al2 = -437.5;\ bt2 = 0.003;\ x0\_1 = 1;\ x0\_2 = 3;\ x0\_3 = 5.5;\ x0\_4 = 10;$
$acc = 10^{-5};\ y0\_1 = -2000;\ y0\_2 = -4000;\ len1 = 40\pi;\ len2 = 70\pi;$
$len3 = -30\pi;\ len4 = -30\pi;\ len5 = 25\pi;\ len6 = 15\pi;\ len7 = -\pi;\ len8 = -\pi$
one can get four limit cycles by the above code.

Information for construction of localization trajectories

	Limit cycle	1 trajectory init. point	time intergation	2 trajectory init. point	time intergation	time direction
1	Left	$(-1000, y0\_1)$	$len7$	$(-1000, y0\_2)$	$len8$	indirect
2	Right smallest	$(x0\_1, 0)$	$len1$	$(x0\_2, 0)$	$len2$	direct
3	Rigth internal	$(x0\_2, 0)$	$len3$	$(x0\_3, 0)$	$len4$	indirect
4	Rigth external	$(x0\_3, 0)$	$len5$	$(x0\_4, 0)$	$len6$	direct

	Limit cycle	1 trajectory init. point	time intergation	2 trajectory init. point	time intergation	time direction
1	Left	$(-1000, -2000)$	$-\pi$	$(-1000, -4000)$	$-\pi$	
2	Right smallest	$(1, 0)$	40π	$(3, 0)$	70π	
3	Rigth internal	$(3, 0)$	-30π	$(5.5, 0)$	-30π	
4	Rigth external	$(5.5, 0)$	25π	$(10, 0)$	15π	