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Hidden oscillations in dynamical systems

G.A. LEONOV[a], N.V. KUZNETSOV[b], O.A. KUZNETSOVA[b], S.M. SELEDZHI[a], V.I.VAGAITSEV[b]
[a]St.Petersburg State University, Universitetsky pr. 28, St.Petersburg, 198504, RUSSIA
[b]University of Jyväskylä, P.O. Box 35 (Agora), FIN-40014, FINLAND

Abstract:- The classical attractors of Lorenz, Rössler, Chua, Chen, and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use *standard numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it.* However there are attractors of another type: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria.* Study of hidden oscillations and attractors requires the development of new analytical and numerical methods which will be considered in this paper.

Key-Words:- Hidden oscillation, attractor localization, hidden attractor, harmonic balance, describing function method, Aizerman conjecture, Kalman conjecture, Hilbert 16th problem

1 Introduction

In the initial period of development of the theory of nonlinear oscillations in the first half of last century [1, 2, 3, 4], a main attention has been given to analysis and synthesis of oscillating systems for which solving the problem of the existence of the oscillation modes did not present any great difficulties. The structure of many mechanical, electromechanical and electronic systems was such that there were oscillation modes in them, the existence of which was almost obvious — oscillations are excited from unstable equilibria. From computational point of view this allows one to use *standard numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it.*

Consider corresponding classical examples.

Example 1 *Van der Pol oscillator*

Consider an oscillations arising in the electrical circuit — the van der Pol oscillator [5]

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

and carry out its simulation for the parameter $\mu = 2$ (see Fig. 1).

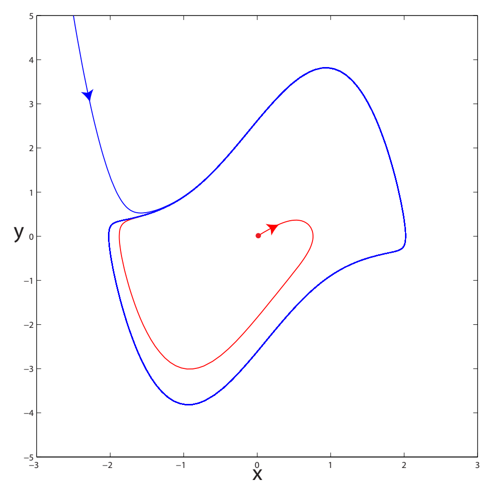

Figure 1: Numerical localization of limit cycle in Van der Pol oscillator

Example 2 *Belousov-Zhabotinsky (BZ) reaction*

In 1951 B.P. Belousov first discovered oscillations in the chemical reactions in liquid phase [6]. Consider one of the Belousov-Zhabotinsky dynamic model

$$\begin{aligned} \varepsilon\dot{x} &= x(1-x) + \frac{f(q-x)}{q+x}z, \\ \dot{z} &= x - z \end{aligned} \tag{2}$$

and carry out its simulation with standard parameters $f = 2/3$, $q = 8 \times 10^{-4}$, $\varepsilon = 4 \times 10^{-2}$ (see Fig. 2).

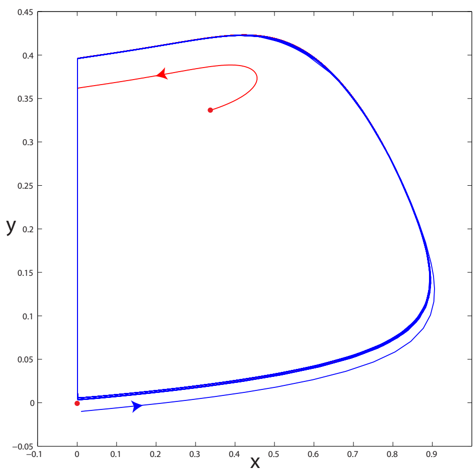

Figure 2: Numerical localization of limit cycle in Belousov-Zhabotinsky (BZ) reaction

Now consider three-dimensional dynamic models.

Example 3 *Lorenz system*

Consider Lorenz system [7]

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= x(\rho - z) - y, \\ \dot{z} &= xy - \beta z \end{aligned} \tag{3}$$

and carry out its simulation with standard parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ (see Fig. 3).

Example 4 *Chua system*

Consider the behavior of the classical Chua circuit [8]. Consider its dynamic model in dimensionless coordinates

$$\begin{aligned} \dot{x} &= \alpha(y - x) - \alpha f(x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -(\beta y + \gamma z). \end{aligned} \tag{4}$$

Here the function

$$f(x) = m_1 x + (m_0 - m_1)\text{sat}(x) \tag{5}$$

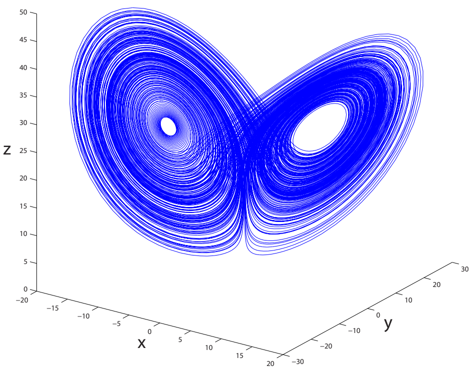

Figure 3: Numerical localization of chaotic attractor in Lorenz system

characterizes a nonlinear element, of the system, called Chuas diode; $\alpha, \beta, \gamma, m_0, m_1$ — are parameters of the system. In this system it was discovered the strange attractors [9] called then Chuas attractors. To date all known classical Chuas attractors are the attractors that are excited from unstable equilibria. This makes it possible to compute different Chuas attractors with relative easy [10, 11, 12, 13, 14, 15]. Here we simulate this system with parameters $\alpha = 9.35$, $\beta = 14.79$, $\gamma = 0.016$, $m_0 = -1.1384$, $m_1 = 0.7225$ (see Fig. 4).

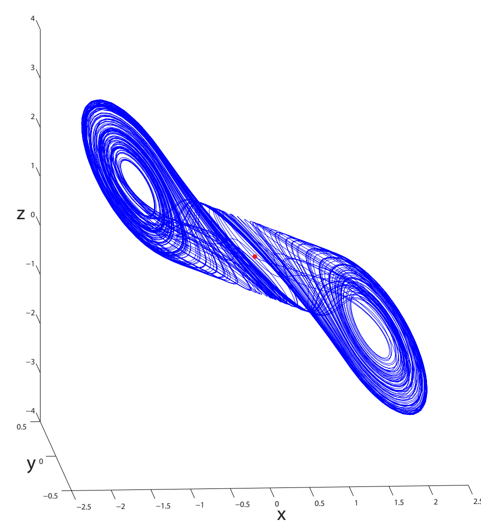

Figure 4: Numerical localization of chaotic attractor in Chua circuit

Here, in all the above examples, limit cycles and attractors are those excited from unstable equilibria.

2 Hidden oscillations and attractors

Further there came to light so called *hidden oscillations* - the oscillations, the existence itself of which is not obvious (which are "small" and, therefore, are difficult for numerical analysis or are not "connected" with equilibrium, i.e. the creation of numerical procedure of integration of trajectories for the passage from equilibrium to periodic solution is impossible). In addition, in this case the integration of trajectories with random initial data is unlikely to furnish the desired result since a basin of attraction can be highly small and the considered system dimension can be large.

For the first time the problem of finding hidden oscillations had been stated by D. Hilbert in 1900 (Hilbert's 16^{th} problem) for two-dimensional polynomial systems. For a more than century history, in the framework of the solution of this problem the numerous theoretical and numerical results were obtained. However the problem is still far from being resolved even for the simple class of quadratic systems. In 40-50s of the 20^{th} century A.N. Kolmogorov became the initiator of a few hundreds of computational experiments [16], in the result of which the limit cycles in two-dimensional quadratic systems would been found. The result was absolutely unexpected: in not a single experiment a limit cycle was found, though it is known that quadratic systems with limit cycles form open domains in the space of coefficients and, therefore, for a random choice of polynomial coefficients, the probability of hitting in these sets is positive. It should be noted also that small and nested cycles [17, 18, 16, 19, 20, 21] are difficult to numerical analysis.

Example 5 *Four limit cycles in quadratic system*

Consider the following quadratic system

$$\begin{aligned} \frac{dx}{dt} &= x^2 + xy + y, \\ \frac{dy}{dt} &= a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y. \end{aligned} \tag{6}$$

Application of special analytical methods [18, 22] allow us to visualize in this system four limit cycle. In Fig. 5 for set of the coefficients $b_2 = 2.7, \quad c_2 = 0.4, \quad a_2 = -10, \quad \alpha_2 = -437.5, \quad \beta_2 = 0.003$ three "large" limit cycles around zero point and 1 "large" limit cycle to the left of straight line $x = -1$ can be observed [23].

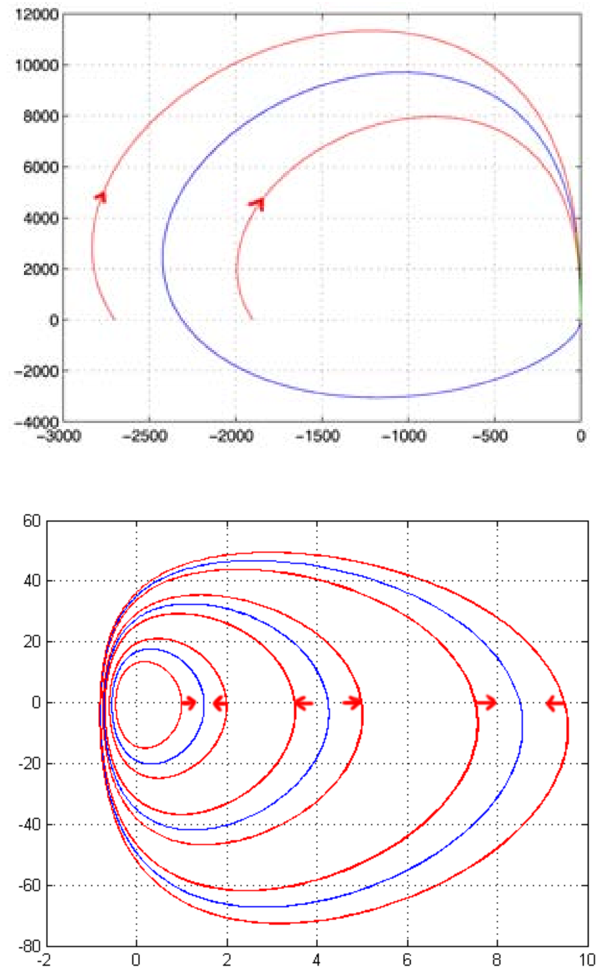

Figure 5: Visualization of 4 limit cycles in quadratic system

Further the problem of analysis of hidden oscillations arose in applied problems of automatic control. In the process of investigation, connected with Aizerman's (1949) and Kalman's (1957) conjectures, it was stated that the differential equations of systems of automatic control, which satisfy generalized Routh-Hurwitz stability criterion, can also have hidden periodic regimes [24].

2.1 Analytical-numerical method for finding hidden oscillations of multidimensional dynamical systems

Consider a system with one scalar[1] nonlinearity

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{7}$$

Here $\mathbf{P}$ is a constant $(n \times n)$-matrix, $\mathbf{q}, \mathbf{r}$ are constant n-dimensional vectors, * is a transposition operation, $\psi(\sigma)$ is a continuous piecewise-differentiable[2] scalar function, and $\psi(0) = 0$. Define a coefficient of harmonic linearization k in such a way that the matrix

$$\mathbf{P}_0 = \mathbf{P} + k\mathbf{q}\mathbf{r}^* \tag{8}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 > 0$) and the rest of its eigenvalues have negative real parts. We assume that such k exists. Rewrite system (7) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x}), \tag{9}$$

where $\varphi(\sigma) = \psi(\sigma) - k\sigma$.

Introduce a finite sequence of functions $\varphi^0(\sigma)$, $\varphi^1(\sigma), \ldots, \varphi^m(\sigma)$ such that the graphs of neighboring functions $\varphi^j(\sigma)$ and $\varphi^{j+1}(\sigma)$ slightly differ from one another, the function $\varphi^0(\sigma)$ is small, and $\varphi^m(\sigma) = \varphi(\sigma)$. Using a smallness of function $\varphi^0(\sigma)$, we can apply and mathematically strictly justify [26, 27, 28, 16, 25, 29] the method of harmonic linearization (describing function method) for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^0(\mathbf{r}^*\mathbf{x}) \tag{10}$$

and determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$. For the localization of oscillating solution (or attractor) of original system (9), we shall follow numerically the transformation of this periodic solution (a starting *oscillating attractor — an attractor, not including equilibria*, denoted further by $\mathcal{A}_0$) with increasing j. Here two cases are possible: all the points of $\mathcal{A}_0$ are in an attraction domain of attractor $\mathcal{A}_1$, being an oscillating attractor of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^j(\mathbf{r}^*\mathbf{x}) \tag{11}$$

with $j = 1$, or in the change from system (10) to system (11) with $j = 1$ it is observed a loss of stability (bifurcation) and the vanishing of $\mathcal{A}_0$. In the first case the solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of system (11) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. If in the process of computation the solution $\mathbf{x}^1(t)$ has not fallen to an equilibrium and it is not increased indefinitely (here a sufficiently large computational interval $[0, T]$ should always be considered), then this solution reaches an attractor $\mathcal{A}_1$. Then it is possible to proceed to system (11) with $j = 2$ and to perform a similar procedure of computation of $\mathcal{A}_2$, by starting a trajectory of system (11) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ and computing the trajectory $\mathbf{x}^2(t)$.

Proceeding this procedure and sequentially increasing j and computing $\mathbf{x}^j(t)$ (being a trajectory of system (11) with initial data $\mathbf{x}^{j-1}(T)$) we either arrive at the computation of $\mathcal{A}_m$ (being an attractor of system (11) with $j = m$, i.e. original system (9)), either, at a certain step, observe a loss of stability (bifurcation) and the vanishing of attractor.

To determine the initial data $\mathbf{x}^0(0)$ of starting periodic solution, system (10) with nonlinearity $\varphi^0(\sigma)$ is transformed by linear nonsingular transformation $\mathbf{S}$ to the form

$$\begin{aligned} \dot{y}_1 &= -\omega_0 y_2 + b_1\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3), \\ \dot{y}_2 &= \omega_0 y_1 + b_2\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3), \\ \dot{\mathbf{y}}_3 &= \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}_3\varphi^0(y_1 + \mathbf{c}_3^*\mathbf{y}_3). \end{aligned} \tag{12}$$

Here y_1, y_2 are scalar values, $\mathbf{y}_3$ is $(n-2)$-dimensional vector; $\mathbf{b}_3$ and $\mathbf{c_3}$ are $(n-2)$-dimensional vectors, b_1 and b_2 are real numbers; $\mathbf{A}_3$ is an $((n-2) \times (n-2))$-matrix, all eigenvalues of which have negative real parts. Without loss of generality, it can be assumed that for the matrix $\mathbf{A}_3$ there exists a positive number $d > 0$ such that

$$\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \le -2d|\mathbf{y}_3|^2, \quad \forall\, \mathbf{y}_3 \in \mathbb{R}^{n-2}. \tag{13}$$

Introduce the describing function

$$\Phi(a) = \int\limits_0^{2\pi/\omega_0} \varphi\big(\cos(\omega_0 t)a\big)\cos(\omega_0 t)dt.$$

In practice, to determine k and ω_0 it is used the transfer function $W(p)$ of system (7):

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q},$$

where p is a complex variable. The number ω_0 is determined from the equation $\mathrm{Im}W(i\omega_0) = 0$ and k is computed then by formula $k = -(\mathrm{Re}W(i\omega_0))^{-1}$.

[1] Vector nonlinearity can be considered similarly [25]

[2] This condition can be weakened if a piecewise-continuous function being Lipschitz on closed continuity intervals is considered [16]

Example 6 *Counterexample to Kalman conjecture*

In 1957 R.E. Kalman formulated the following conjecture [30]: *Suppose that for all $k \in (\mu_1, \mu_2)$ a zero solution of system* (9) *with $\varphi(\sigma) = k\sigma$ is asymptotically stable in the large (i.e., a zero solution is Lyapunov stable and any solution of system* (9) *tends to zero as $t \to \infty$. In other words, a zero solution is a global attractor of system* (9) *with $\varphi(\sigma) = k\sigma$).*

If at the points of differentiability of $\varphi(\sigma)$ the condition

$$\mu_1 < \varphi'(\sigma) < \mu_2 \qquad (14)$$

is satisfied, then system (9) *is stable in the large?*

Consider a method for counterexamples construction Kalman's conjecture. Let us assume first that $\mu_1 = 0, \mu_2 > 0$ and consider system (12) with nonlinearity $\varphi^0(\sigma)$ of special form

$$\varphi^0(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon; \\ \text{sign}(\sigma)M\varepsilon^3, & \forall|\sigma| > \varepsilon. \end{cases} \qquad (15)$$

Here $\mu < \mu_2$ and M are certain positive numbers, ε is a small positive parameter.

Then the following result is valid.

Theorem 1 *[31] If the inequalities*

$$b_1 < 0,$$
$$0 < \mu b_2\omega_0(\mathbf{c_3}^*\mathbf{b_3} + b_1) + b_1\omega_0^2$$

are satisfied, then for small enough ε system (12) with nonlinearity (15) *has orbitally stable periodic solution, satisfying the following relations*

$$\begin{aligned} & y_1(t) = -\sin(\omega_0 t)x_2(0) + O(\varepsilon), \\ & y_2(t) = \cos(\omega_0 t)x_2(0) + O(\varepsilon), \\ & \mathbf{y}_3(t) = \mathbf{O}_{n-2}(\varepsilon), \\ & y_1(0) = O(\varepsilon^2), \\ & y_2(0) = -\sqrt{\frac{\mu(\mu b_2\omega_0(\mathbf{c_3}^*\mathbf{b_3} + b_1) + b_1\omega_0^2)}{-3\omega_0^2 M b_1}} + O(\varepsilon), \\ & \mathbf{y}_3(0) = \mathbf{O}_{n-2}(\varepsilon^2). \end{aligned} \qquad (16)$$

The methods for the proof of this theorem are developed in [26, 16, 32, 33].

Based on this theorem, it is possible to apply described above multi-step procedure for the localization of hidden oscillations: initial data obtained in this theorem allow to step aside from stable zero equilibrium and to start numerical localization of possible oscillations.

For that we consider a finite sequence of piecewise-linear functions

$$\varphi^j(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon_j; \\ \text{sign}(\sigma)M\varepsilon_j^3, & \forall|\sigma| > \varepsilon_j. \end{cases}, \quad \varepsilon_j = \tfrac{j}{m}\sqrt{\tfrac{\mu}{M}}$$
$$j = 1, \dots, m. \qquad (17)$$

Here function $\varphi^m(\sigma)$ is monotone continuous piecewise-linear function $\text{sat}(\sigma)$ ("saturation"). We choose m in such a way that the graphs of functions φ^j and φ^{j+1} are slightly distinct from each other outside small neighborhoods of points of discontinuity.

Suppose that the periodic solution $\mathbf{x}^m(t)$ of system (9) with monotone and continuous function $\varphi^m(\sigma)$ ("saturation") is computed. In this case we organize a similar computational procedure for the sequence of systems

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}x + \mathbf{q}\psi^i(\mathbf{r}^*\mathbf{x}). \qquad (18)$$

Here $i = 0, \dots, h, \quad \psi^0(\sigma) = \varphi^m(\sigma)$ and

$$\psi^i(\sigma) = \varphi^m(\sigma) + \begin{cases} 0, & \forall|\sigma| \le \varepsilon_m; \\ i(\sigma - \text{sign}(\sigma)\varepsilon_m)N, & \forall|\sigma| > \varepsilon_m, \end{cases}$$

where N is a certain positive parameter such that $hN < \mu_2$ (using the technique of small changes, it is also possible to approach other continuous monotonic increasing functions [25]).

The finding of periodic solutions $\mathbf{x}^i(t)$ of system (18) gives a certain counterexample to Kalman's conjecture for each $i = 1, \dots, h$.

Consider a system

$$\begin{aligned} \dot{x}_1 &= -x_2 - 10\,\varphi(x_1 - 10.1\,x_3 - 0.1\,x_4), \\ \dot{x}_2 &= x_1 - 10.1\,\varphi(x_1 - 10.1\,x_3 - 0.1\,x_4), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -x_3 - x_4 + \varphi(x_1 - 10.1\,x_3 - 0.1\,x_4). \end{aligned} \qquad (19)$$

Here for $\varphi(\sigma) = k\sigma$ linear system (19) is stable for $k \in (0, 9.9)$ and by the above-mentioned theorem for piecewise-continuous nonlinearity $\varphi(\sigma) = \varphi^0(\sigma)$ with sufficiently small ε there exists periodic solution.

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Now we make use of the algorithm of constructing of periodic solutions. Suppose $\mu = M = 1$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.2$, ..., $\varepsilon_{10} = 1$. For $j = 1, ..., 10$, we construct sequentially solutions of system (19), assuming that by (17) the nonlinearity $\varphi(\sigma)$ is equal to $\varphi^j(\sigma)$. Here for all ε_j, $j = 1, ..., 10$ there exists periodic solution.

At the first step for $j = 0$ by the theorem the initial data of stable periodic oscillation take the form:

$$x_1(0) = O(\varepsilon),\ x_3(0) = O(\varepsilon),\ x_4(0) = O(\varepsilon),$$
$$x_2(0) = -1.7513 + O(\varepsilon).$$

Therefore for $j = 1$ a trajectory starts from the point $x_1(0) = x_3(0) = x_4(0) = 0$, $x_2(0) = -1.7513$. The projection of this trajectory on the plane (x_1, x_2) and the output of system $\mathbf{r}^*\mathbf{x}(t) = x_1(t) - 10.1x_3(t) - 0.1x_4(t)$ are shown in Fig. 6.

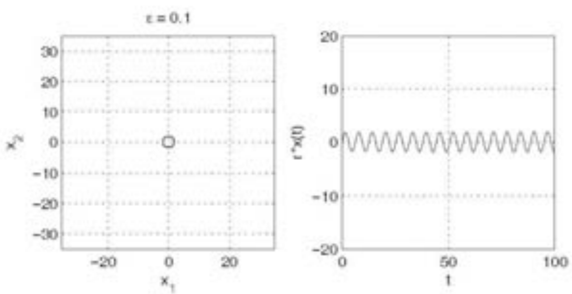


Figure 6: $\varepsilon_1 = 0.1$: trajectory projection on the plane (x_1, x_2)

From the figure it follows that after transient process stable periodic solution is reached. At the first step the computational procedure is ended at the point $x_1(T) = 0.7945$, $x_2(T) = 1.7846$, $x_3(T) = 0.0018$, $x_4(T) = -0.0002$, where $T = 1000\pi$.

Further, for $j = 2$ we take the following initial data: $x_1(0) = 0.7945$, $x_2(0) = 1.7846$, $x_3(0) = 0.0018$, $x_4(0) = -0.0002$, and obtain next periodic solutions.

Proceeding this procedure for $j = 3, ...10$, we sequentially approximate (Fig. 8-14) a periodic solution of system (19) (Fig. 15).

Note that for $\varepsilon_j = 1$ the nonlinearity $\varphi^j(\sigma)$ is monotone. The computational process is ended at the point $x_1(T) = 1.6193$, $x_2(T) = -29.7162$, $x_3(T) = -0.2529$, $x_4(T) = 1.2179$, where $T = 1000\pi$.

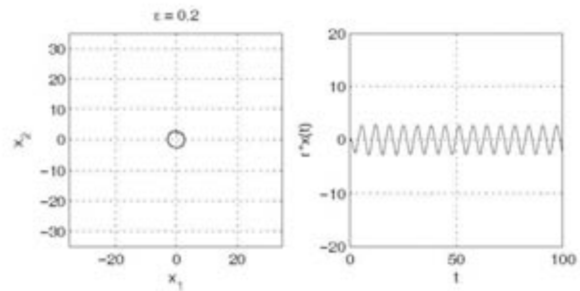


Figure 7: $\varepsilon_2 = 0.2$: trajectory projection on the plane (x_1, x_2)

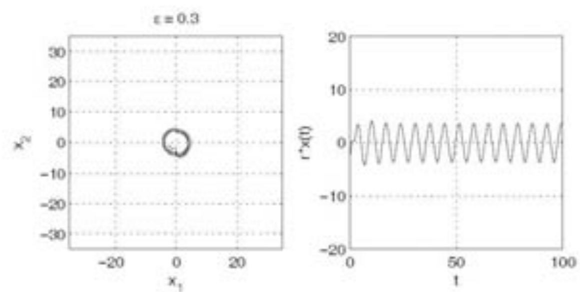


Figure 8: $\varepsilon_3 = 0.3$: trajectory projection on the plane (x_1, x_2)

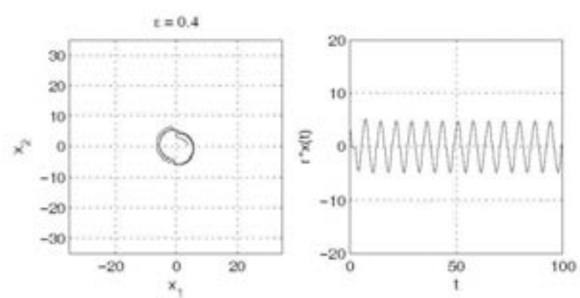


Figure 9: $\varepsilon_4 = 0.4$: trajectory projection on the plane (x_1, x_2)

We also remark that here if instead of sequential increasing of ε_j, we compute a solution with initial data according to (16) for $\varepsilon = 1$, then the solution will "falls down" to zero.

Change the nonlinearity $\varphi(\sigma)$ to the strictly increasing function $\psi^i(\sigma)$, where $\mu = 1, \varepsilon_m = 1$, $N = 0.01$, for i=1,...,5, and, continue the sequential con-

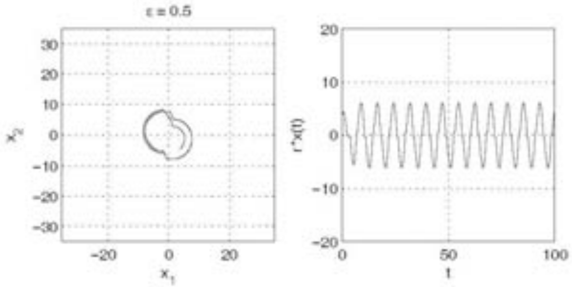


Figure 10: $\varepsilon_5 = 0.5$: trajectory projection on the plane (x_1, x_2)

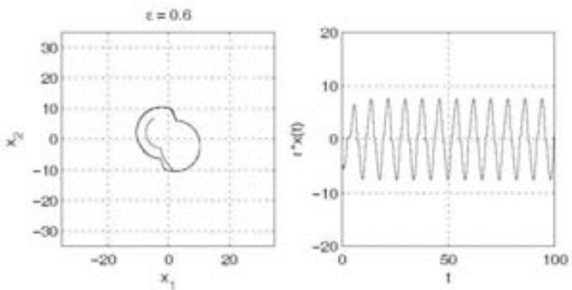


Figure 11: $\varepsilon_6 = 0.6$: trajectory projection on the plane (x_1, x_2)

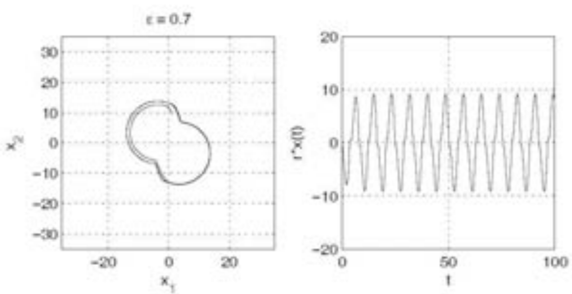


Figure 12: $\varepsilon_7 = 0.7$: trajectory projection on the plane (x_1, x_2)

struction of periodic solutions for system (19). The graph of such nonlinearity is shown in Fig. 16.

The periodic solutions obtained are shown in Fig. 17–21.

In the case of the computation of solution for

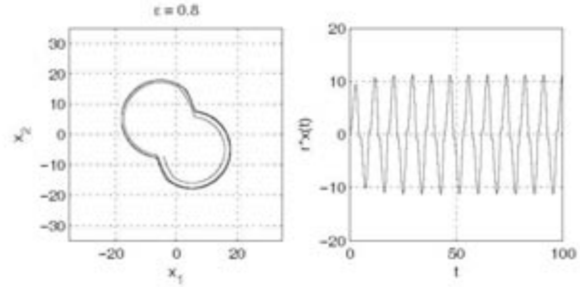


Figure 13: $\varepsilon_8 = 0.8$: trajectory projection on the plane (x_1, x_2)

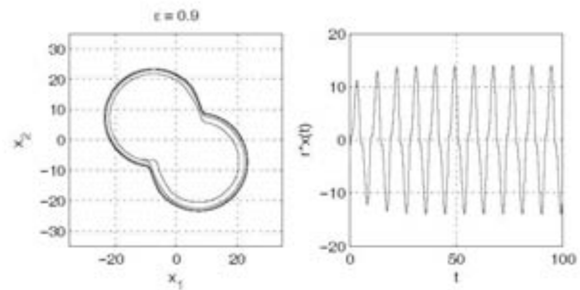


Figure 14: $\varepsilon_9 = 0.9$: trajectory projection on the plane (x_1, x_2)

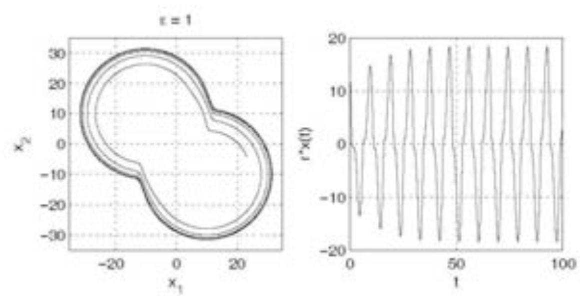


Figure 15: $\varepsilon_{10} = 1$: trajectory projection on the plane (x_1, x_2)

$i = 6$ there occurs the vanishing of periodic solution (Fig. 22).

For system (19) with smooth strictly increasing nonlinearity

$$\varphi(\sigma) = \tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}} \qquad (20)$$

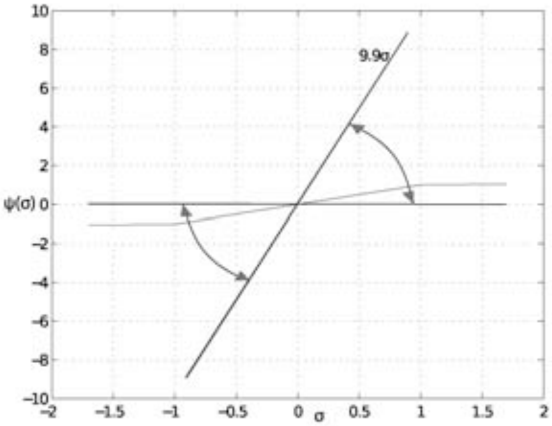


Figure 16: The graph of $\psi^i(\sigma)$ for $i = 5$ and stability sector

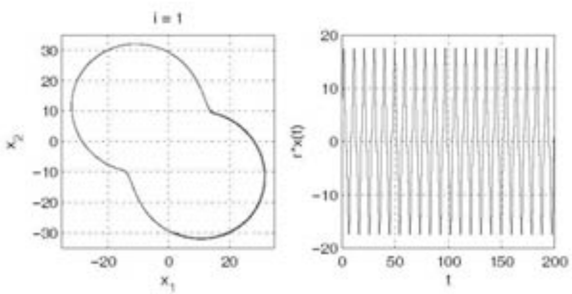


Figure 17: $i = 1$: trajectory projection on the plane (x_1, x_2)

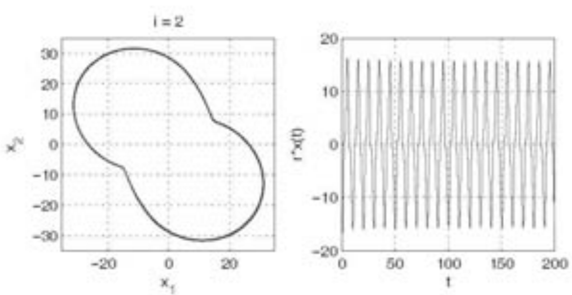


Figure 18: $i = 2$: trajectory projection on the plane (x_1, x_2)

there exists a periodic solution (Fig. 23). Here

$$0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1, \ \forall\sigma.$$

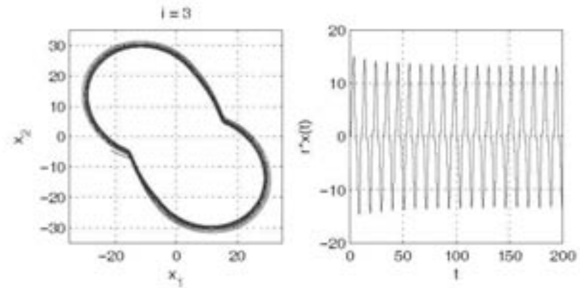


Figure 19: $i = 3$: trajectory projection on the plane (x_1, x_2)

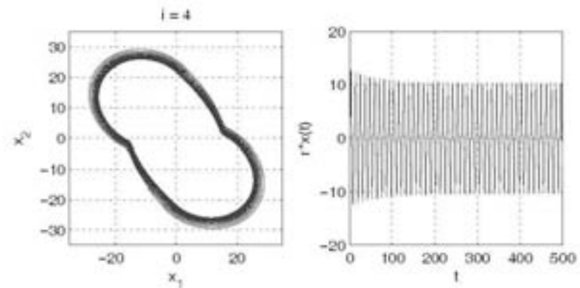


Figure 20: $i = 4$: trajectory projection on the plane (x_1, x_2)

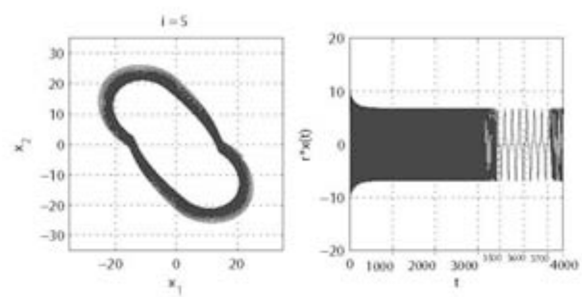


Figure 21: $i = 5$: trajectory projection on the plane (x_1, x_2)

Here on the first step it is possible to apply described above method to reach saturation function; on the second — "slightly" by small steps transform saturation to tanh.

Further, the issues analysis of hidden oscillations arose in the study of dynamical phase locked loops

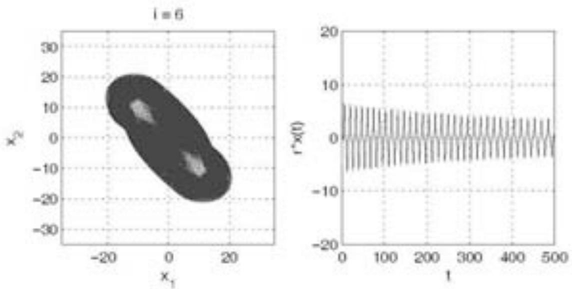


Figure 22: $i = 6$: trajectory projection on the plane (x_1, x_2)

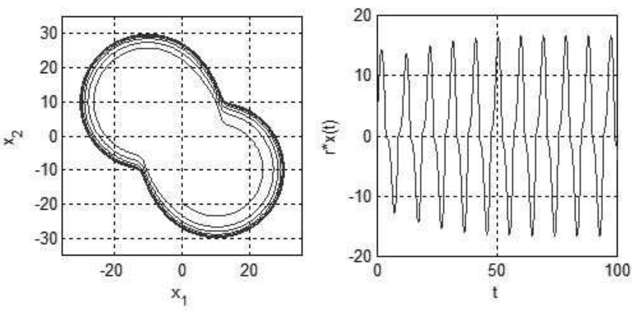


Figure 23: The projection of trajectory with the initial data $x_1(0) = x_3(0) = x_4(0) = 0,\ x_2(0) = -20$ of system (20) on the plane (x_1, x_2)

[34,35,36]. In 1961 semi-stable limit cycle were founded in two-dimensional model of PLL [37] (which also can not be detected by numerical simulations).

Example 7 *Hidden attractor in Chua's circuit*

Similar situation arises in attractors localization. The classical attractors of Lorenz, Rossler, Chua, Chen, and other widely-known attractors are those excited from unstable equilibria. However there are attractors of another type [38]: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria*. Numerical localization, computation, and analytical investigation of such attractors are much more difficult problems.

Recently such hidden attractors were discovered [38, 39] in classical Chua's circuit.

Here we consider system (10) with $\varphi^0(\sigma) = \varepsilon\varphi(\sigma)$ were ε is a small positive parameter and introduce class of functions φ^j: $\varphi^1 = \varepsilon_1\varphi(\sigma)$, …, $\varphi^{m-1} = \varepsilon_{m-1}\varphi(\sigma)$, $\varphi^m = \varepsilon_m\varphi(\sigma)$, where $\varepsilon_j = j/m$, $j = 1, \ldots, m$.

For such class on nonlinearities φ_j the following theorem was proved in [16]

Theorem 2 *If it can be found a positive a_0 such that*

$$\Phi(a_0) = 0, \tag{21}$$

then for the initial data of periodic solution $\mathbf{x}^0(0) = \mathbf{S}(y_1(0), y_2(0), \mathbf{y}_3(0))^*$ at the first step of algorithm we have

$$y_1(0) = a_0 + O(\varepsilon),\ y_2(0) = 0,\ \mathbf{y}_3(0) = \mathbf{O_{n-2}}(\varepsilon), \tag{22}$$

where $\mathbf{O_{n-2}}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$.

For the stability of $\mathbf{x}^0(t)$ (if the stability is regarded in the sense that for all solutions with the initial data sufficiently close to $\mathbf{x}^0(0)$ the modulus of their difference with $\mathbf{x}^0(t)$ is uniformly bounded for all $t > 0$), it is sufficient to require the satisfaction of the following condition

$$b_1 \frac{d\Phi(a)}{da}\bigg|_{a=a_0} < 0.$$

Consider Chua system (4) with the parameters

$$\begin{aligned} &\alpha = 8.4562,\ \beta = 12.0732,\ \gamma = 0.0052,\\ &m_0 = -0.1768,\ m_1 = -1.1468. \end{aligned} \tag{23}$$

Note that for the considered values of parameters there are three equilibria in the system: a locally stable zero equilibrium and two saddle equilibria.

Modeling of this system was carried out in 10 steps increasing parameter ε_j from the start value $\varepsilon_1 = 0.1$ to the finish value $\varepsilon_{10} = 1$ with step 0.1. Projections of trajectories of the system into (x, y) plane at the each step of the multistage numerical procedure described above are shown in Figs. 24—33.

Here application of special analytical-numerical algorithm described above [25] allow us to find hidden attractor — $\mathcal{A}_{\text{hidden}}$ (see Fig. 34).

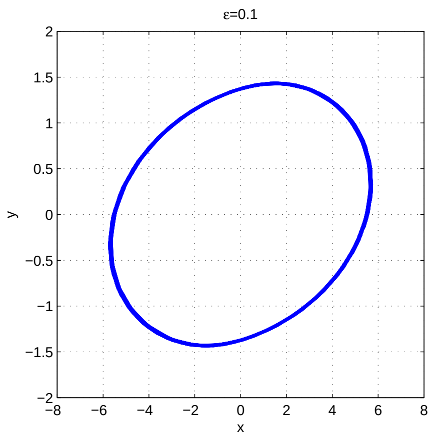


Figure 24: Projections of trajectory into (x, y) plane for $\varepsilon_1 = 0.1$

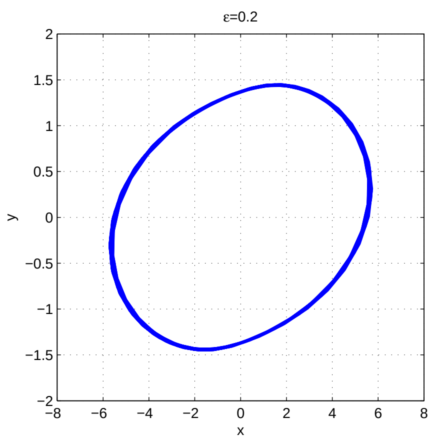


Figure 25: $\varepsilon_2 = 0.2$

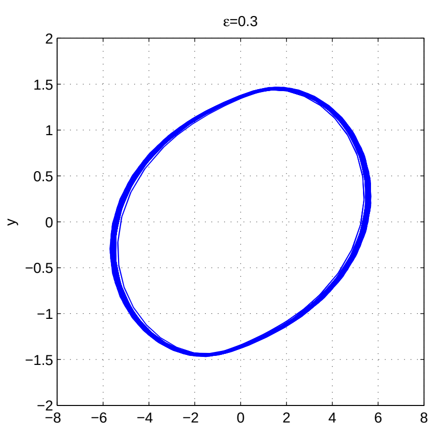


Figure 26: $\varepsilon_3 = 0.3$

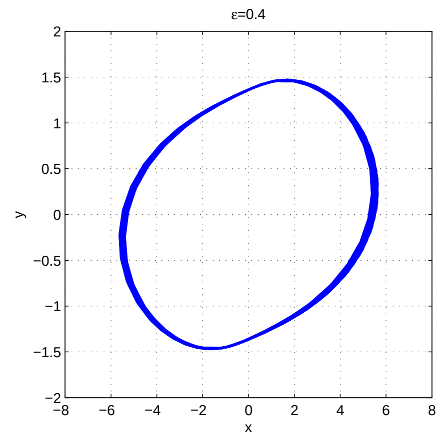


Figure 27: $\varepsilon_4 = 0.4$

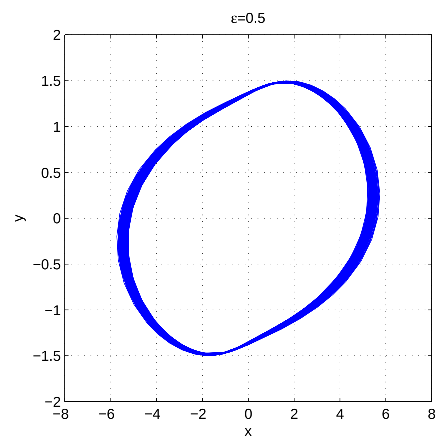


Figure 28: $\varepsilon_5 = 0.5$

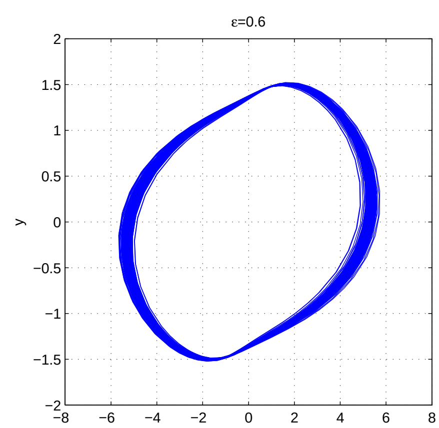


Figure 29: $\varepsilon_6 = 0.6$

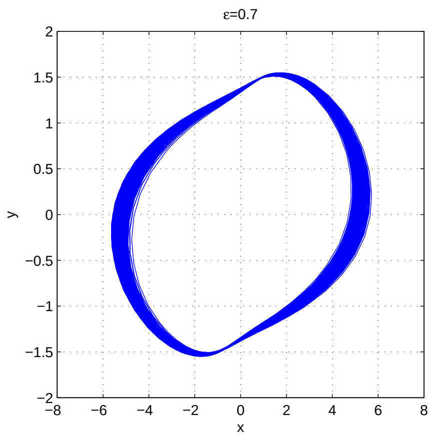


Figure 30: $\varepsilon_7 = 0.7$

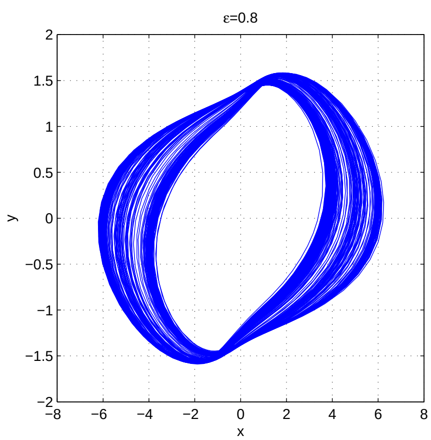


Figure 31: $\varepsilon_8 = 0.8$

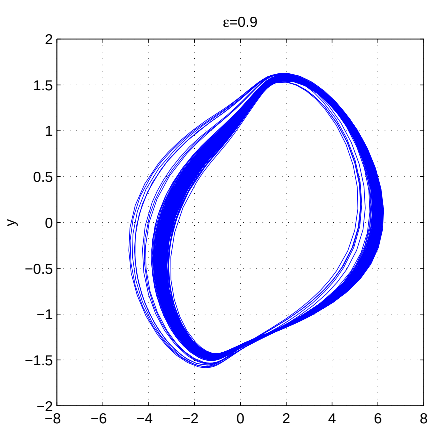


Figure 32: $\varepsilon_9 = 0.9$

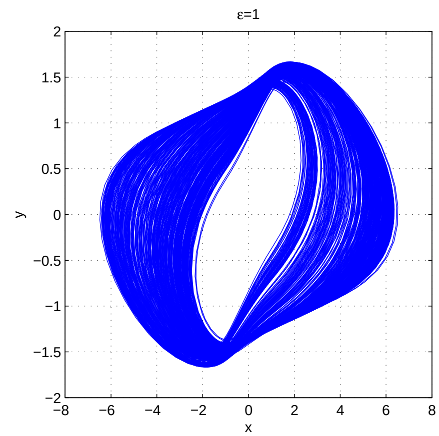
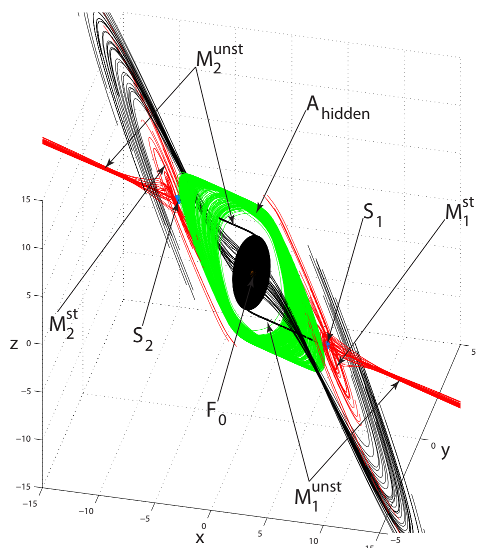


Figure 33: Projections of trajectory into (x, y) plane for $\varepsilon_{10} = 1$

It should be noted that the decreasing of integration step, the increasing of integration time, and the computation of different trajectories of original system with initial data from a small neighborhood of $\mathcal{A}_{\text{hidden}}$ lead to the localization of the same set $\mathcal{A}_{\text{hidden}}$ (all the computed trajectories densely trace the set $\mathcal{A}_{\text{hidden}}$). Note also that for the computed trajectories it is observed Zhukovsky instability and the positiveness of Lyapunov exponent [40, 41].



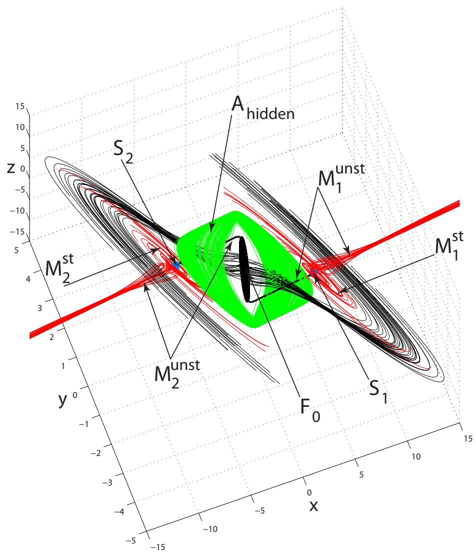


Figure 34: Equilibrium, stable manifolds of saddles, and localization of hidden attractor.

The behavior of system trajectories in the neighborhood of equilibria is shown in Fig. 34. Here $M_{1,2}^{\mathrm{unst}}$ are unstable manifolds, $M_{1,2}^{\mathrm{st}}$ are stable manifolds. Thus, in a phase space of system there are stable separating manifolds of saddles.

The above and the remark on the existence, in system, of locally stable zero equilibrium F_0 attracted the stable manifolds $M_{1,2}^{\mathrm{st}}$ of two symmetric saddles S_1 and S_2 led to the conclusion that in $\mathcal{A}_{\mathrm{hidden}}$ there is computed a hidden strange attractor.

3 Conclusion

Study of hidden oscillations and attractors requires the development of new analytical and numerical methods. At this invited lecture there are discussed the new analytic-numerical approaches to investigation of hidden oscillations in dynamical systems, based on the development of numerical methods, computers, and applied bifurcation theory, which suggests revisiting and revising early ideas on the application of the small parameter method and the harmonic linearization [16, 25, 29, 38].

Acknowledgements

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