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# Localization of hidden Chua's attractors

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**Abstract.** The classical attractors of Lorenz, Rossler, Chua, Chen, and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it. However there are attractors of another type: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria.* In the present Letter for localization of hidden attractors of Chua's circuit it is suggested to use a special analytical-numerical algorithm.

**Keywords:** chaotic hidden attractor, Chua system, Chua circuits, hidden oscillation, describing function method

## 1 Introduction

The classical attractors of Lorenz [1], Rossler [2], Chua [3], Chen [4], and other widely-known attractors are those excited from unstable equilibria. From computational point of view this allows one to use numerical method, in which after transient process a trajectory, started from a point of unstable manifold in the neighborhood of equilibrium, reaches an attractor and identifies it. However there are attractors of another type: *hidden attractors, a basin of attraction of which does not contain neighborhoods of equilibria.* The simplest examples of systems with such attractors are counterexamples to widely-known Aizerman's and Kalman's conjectures on absolute stability (see, e.g.,[8, 10]). Numerical localization, computation, and analytical investigation of such attractors are much more difficult problems. In the present Letter for localization of hidden attractors of Chua's circuit it is suggested to use a special analytical-numerical algorithm.

Chua's circuit can be described by differential equations in dimensionless coordinates:

$$\begin{aligned} \dot{x} &= \alpha(y-x) - \alpha f(x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -(\beta y + \gamma z). \end{aligned} \tag{1.1}$$

Here the function

$$f(x) = m_1 x + (m_0 - m_1)\mathrm{sat}(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| - |x-1|) \tag{1.2}$$

characterizes a nonlinear element, of the system, called *Chua's diode*; $\alpha, \beta, \gamma, m_0, m_1$ are parameters of the system. In this system it was discovered the strange attractors [11, 12] called then Chua's attractors (for the current state of chaotic behavior investigation in Chua's circuit see, e.g., recent work [13] and references within).

To date all known Chua's attractors are the attractors that are excited from unstable equilibria. This makes it possible to compute different Chua's attractors [14, 15, 16] with relative easy.

The applied in this Letter algorithm shows for the first time the possibility of existence of hidden attractor in system (1.1). Note that L. Chua himself, analyzing in the work [3] different cases of attractor existence in Chua's circuit, does not admit the existence of such hidden attractor.

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[2]PDF slides http://www.math.spbu.ru/user/nk/PDF/Hidden-attractor-localization-Chua-circuit.pdf

## 2 Analytical-numerical method for attractors localization

Consider a system with one scalar[3] nonlinearity

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.3}$$

Here $\mathbf{P}$ is a constant $(n \times n)$-matrix, $\mathbf{q}, \mathbf{r}$ are constant $n$-dimensional vectors, $^*$ is a transposition operation, $\psi(\sigma)$ is a continuous piecewise-differentiable[4] scalar function, and $\psi(0) = 0$. Define a coefficient of harmonic linearization $k$ in such a way that the matrix

$$\mathbf{P}_0 = \mathbf{P} + k\mathbf{q}\mathbf{r}^* \tag{2.4}$$

has a pair of purely imaginary eigenvalues $\pm i\omega_0$ $(\omega_0 > 0)$ and the rest of its eigenvalues have negative real parts. We assume that such $k$ exists. Rewrite system (2.3) as

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x}), \tag{2.5}$$

where $\varphi(\sigma) = \psi(\sigma) - k\sigma$.

Introduce a finite sequence of functions $\varphi^0(\sigma), \varphi^1(\sigma), \ldots, \varphi^m(\sigma)$ such that the graphs of neighboring functions $\varphi^j(\sigma)$ and $\varphi^{j+1}(\sigma)$ slightly differ from one another, the function $\varphi^0(\sigma)$ is small, and $\varphi^m(\sigma) = \varphi(\sigma)$. Using a smallness of function $\varphi^0(\sigma)$, we can apply and mathematically strictly justify [5, 6, 7, 8, 9, 10] the method of harmonic linearization (describing function method) for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^0(\mathbf{r}^*\mathbf{x}) \tag{2.6}$$

and determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$. For the localization of attractor of original system (2.5), we shall follow numerically the transformation of this periodic solution (a starting *oscillating attractor* — an attractor, not including equilibria, denoted further by $\mathcal{A}_0$) with increasing $j$. Here two cases are possible: all the points of $\mathcal{A}_0$ are in an attraction domain of attractor $\mathcal{A}_1$, being an oscillating attractor of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varphi^j(\mathbf{r}^*\mathbf{x}) \tag{2.7}$$

with $j = 1$, or in the change from system (2.6) to system (2.7) with $j = 1$ it is observed a loss of stability (bifurcation) and the vanishing of $\mathcal{A}_0$. In the first case the solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of system (2.7) with $j = 1$ from the initial point $\mathbf{x}^0(0)$. If in the process of computation the solution $\mathbf{x}^1(t)$ has not fallen to an equilibrium and it is not increased indefinitely (here a sufficiently large computational interval $[0, T]$ should always be considered), then this solution reaches an attractor $\mathcal{A}_1$. Then it is possible to proceed to system (2.7) with $j = 2$ and to perform a similar procedure of computation of $\mathcal{A}_2$, by starting a trajectory of system (2.7) with $j = 2$ from the initial point $\mathbf{x}^1(T)$ and computing the trajectory $\mathbf{x}^2(t)$.

Proceeding this procedure and sequentially increasing $j$ and computing $\mathbf{x}^j(t)$ (being a trajectory of system (2.7) with initial data $\mathbf{x}^{j-1}(T)$) we either arrive at the computation of $\mathcal{A}_m$ (being an attractor of system (2.7) with $j = m$, i.e. original system (2.5)), either, at a certain step, observe a loss of stability (bifurcation) and the vanishing of attractor.

[3]The case of vector nonlinearity can be considered similarly [9]

[4]This condition can be weakened if a piecewise-continuous function being Lipschitz on closed continuity intervals is considered [8]

To determine the initial data $\mathbf{x}^0(0)$ of starting periodic solution, system (2.6) with nonlinearity $\varphi^0(\sigma)$ is transformed by linear nonsingular transformation $\mathbf{S}$ to the form

$$\begin{aligned} \dot{y}_1 &= -\omega_0 y_2 + b_1 \varphi^0(y_1 + \mathbf{c}_3^* \mathbf{y}_3), \\ \dot{y}_2 &= \omega_0 y_1 + b_2 \varphi^0(y_1 + \mathbf{c}_3^* \mathbf{y}_3), \\ \dot{\mathbf{y}}_3 &= \mathbf{A}_3 \mathbf{y}_3 + \mathbf{b}_3 \varphi^0(y_1 + \mathbf{c}_3^* \mathbf{y}_3). \end{aligned} \tag{2.8}$$

Here $y_1$, $y_2$ are scalar values, $\mathbf{y}_3$ is $(n-2)$-dimensional vector; $\mathbf{b}_3$ and $\mathbf{c_3}$ are $(n-2)$-dimensional vectors, $b_1$ and $b_2$ are real numbers; $\mathbf{A}_3$ is an $((n-2)\times(n-2))$-matrix, all eigenvalues of which have negative real parts. Without loss of generality, it can be assumed that for the matrix $\mathbf{A}_3$ there exists a positive number $d > 0$ such that

$$\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \le -2d|\mathbf{y}_3|^2, \quad \forall\, \mathbf{y}_3 \in \mathbb{R}^{n-2}. \tag{2.9}$$

Introduce the describing function

$$\Phi(a) = \int_0^{2\pi/\omega_0} \varphi\big(\cos(\omega_0 t)a\big)\cos(\omega_0 t)dt.$$

**Theorem 1** *[8] If it can be found a positive $a_0$ such that*

$$\Phi(a_0) = 0, \tag{2.10}$$

*then for the initial data of periodic solution $\mathbf{x}^0(0) = \mathbf{S}(y_1(0), y_2(0), \mathbf{y}_3(0))^*$ at the first step of algorithm we have*

$$y_1(0) = a_0 + O(\varepsilon),\ y_2(0) = 0,\ \mathbf{y}_3(0) = \mathbf{O_{n-2}}(\varepsilon), \tag{2.11}$$

*where $\mathbf{O_{n-2}}(\varepsilon)$ is an $(n-2)$-dimensional vector such that all its components are $O(\varepsilon)$.*

For the stability of $\mathbf{x}^0(t)$ (if the stability is regarded in the sense that for all solutions with the initial data sufficiently close to $\mathbf{x}^0(0)$ the modulus of their difference with $\mathbf{x}^0(t)$ is uniformly bounded for all $t > 0$), it is sufficient to require the satisfaction of the following condition

$$b_1 \frac{d\Phi(a)}{da}\bigg|_{a=a_0} < 0.$$

In practice, to determine $k$ and $\omega_0$ it is used the transfer function $W(p)$ of system (2.3):

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q},$$

where $p$ is a complex variable. The number $\omega_0$ is determined from the equation $\mathrm{Im}W(i\omega_0) = 0$ and $k$ is computed then by formula $k = -(\mathrm{Re}W(i\omega_0))^{-1}$.

# 3 Localization of hidden attractor in Chua's system.

We now apply the above algorithm to analysis of Chua's system. For this purpose, rewrite Chua's system (1.1) in the form (2.3)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{3.12}$$

Here

$$\mathbf{P} = \begin{pmatrix} -\alpha(m_1+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \ \mathbf{q} = \begin{pmatrix} -\alpha \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\psi(\sigma) = (m_0 - m_1)\mathrm{sat}(\sigma).$$

Introduce the coefficient $k$ and small parameter $\varepsilon$, and represent system (3.12) as (2.6)

$$\frac{d\mathbf{x}}{dt} = \mathbf{P_0}\mathbf{x} + \mathbf{q}\varepsilon\varphi(\mathbf{r}^*\mathbf{x}), \tag{3.13}$$

where

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^* = \begin{pmatrix} -\alpha(m_1+1+k) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \ \lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega_0, \ \lambda_3^{\mathbf{P}_0} = -d,$$
$$\varphi(\sigma) = \psi(\sigma) - k\sigma = (m_0 - m_1)\mathrm{sat}(\sigma) - k\sigma.$$

By nonsingular linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ system (3.13) is reduced to the form (2.8)

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} + \mathbf{b}\varepsilon\varphi(\mathbf{c}^*\mathbf{y}), \tag{3.14}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ 1 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}.$$

The transfer function $W_{\mathbf{A}}(p)$ of system (3.14) can be represented as

$$W_{\mathbf{A}}(p) = \frac{-b_1 p + b_2\omega_0}{p^2 + \omega_0^2} + \frac{h}{p+d}.$$

Further, using the equality of transfer functions of systems (3.13) and (3.14), we obtain

$$W_{\mathbf{A}}(p) = \mathbf{r}^*(\mathbf{P}_0 - p\mathbf{I})^{-1}\mathbf{q}.$$

This implies the following relations

$$\begin{aligned}
k &= \frac{-\alpha(m_1 + m_1\gamma + \gamma) + \omega_0^2 - \gamma - \beta}{\alpha(1+\gamma)}, \\
d &= \frac{\alpha + \omega_0^2 - \beta + 1 + \gamma + \gamma^2}{1+\gamma}, \\
h &= \frac{\alpha(\gamma + \beta - (1+\gamma)d + d^2)}{\omega_0^2 + d^2}, \\
b_1 &= \frac{\alpha(\gamma + \beta - \omega_0^2 - (1+\gamma)d)}{\omega_0^2 + d^2}, \\
b_2 &= \frac{\alpha\big((1+\gamma-d)\omega_0^2 + (\gamma+\beta)d\big)}{\omega_0(\omega_0^2 + d^2)}.
\end{aligned} \tag{3.15}$$

Since system (3.13) can be reduced to the form (3.14) by the nonsingular linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$, for the matrix $\mathbf{S}$ the following relations

$$\mathbf{A} = \mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}, \quad \mathbf{b} = \mathbf{S}^{-1}\mathbf{q}, \quad \mathbf{c}^* = \mathbf{r}^*\mathbf{S} \tag{3.16}$$

are valid. Having solved these matrix equations, we obtain the transformation matrix

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

Here

$$s_{11} = 1, \quad s_{12} = 0, \quad s_{13} = -h,$$
$$s_{21} = m_1 + 1 + k, \quad s_{22} = -\frac{\omega_0}{\alpha}, \quad s_{23} = -\frac{h(\alpha(m_1 + 1 + k) - d)}{\alpha},$$
$$s_{31} = \frac{\alpha(m_1 + k) - \omega_0^2}{\alpha}, \quad s_{32} = -\frac{\alpha(\beta + \gamma)(m_1 + k) + \alpha\beta - \gamma\omega_0^2}{\alpha\omega_0},$$
$$s_{33} = h\frac{\alpha(m_1 + k)(d - 1) + d(1 + \alpha - d)}{\alpha}.$$

By (2.11), for small enough $\varepsilon$ we determine initial data for the first step of multistage localization procedure

$$\mathbf{x}(0) = \mathbf{S}\mathbf{y}(0) = \mathbf{S}\begin{pmatrix} a_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_0 s_{11} \\ a_0 s_{21} \\ a_0 s_{31} \end{pmatrix}.$$

Returning to Chua's system denotations, for determining the initial data of starting solution of multistage procedure we have the following formulas

$$x(0) = a_0, \ y(0) = a_0(m_1 + 1 + k), \ z(0) = a_0\frac{\alpha(m_1 + k) - \omega_0^2}{\alpha}. \tag{3.17}$$

Consider system (3.13) with the parameters

$$\alpha = 8.4562, \ \beta = 12.0732, \ \gamma = 0.0052, \ m_0 = -0.1768, \ m_1 = -1.1468. \tag{3.18}$$

Note that for the considered values of parameters there are three equilibria in the system: a locally stable zero equilibrium and two saddle equilibria.

Now we apply the above procedure of hidden attractors localization to Chua's system (3.12) with parameters (3.18). For this purpose, compute a starting frequency and a coefficient of harmonic linearization. We have

$$\omega_0 = 2.0392, \quad k = 0.2098\,.$$

Then, compute solutions of system (3.13) with nonlinearity $\varepsilon\varphi(x) = \varepsilon(\psi(x) - kx)$, sequentially increasing $\varepsilon$ from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$ with the step 0.1.

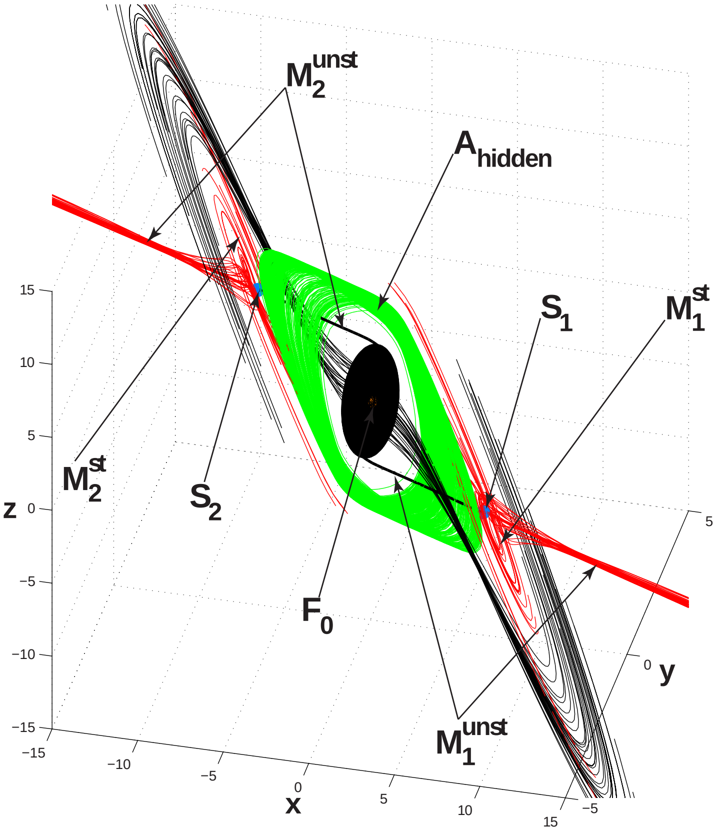
By (3.15) and (3.17) we obtain the initial data

$$x(0) = 9.4287, y(0) = 0.5945, z(0) = -13.4705$$

for the first step of multistage procedure for the construction of solutions. For the value of parameter $\varepsilon_1 = 0.1$, after transient process the computational procedure reaches the starting oscillation $\mathbf{x}^1(t)$. Further, by the sequential transformation $\mathbf{x}^j(t)$ with increasing the parameter $\varepsilon_j$, using the numerical procedure, for original Chua's system (3.12) the set $\mathcal{A}_{\text{hidden}}$ is computed. This set is shown in Fig. 1.

It should be noted that here the decreasing of integration step, the increasing of integration time, and the computation of different trajectories of original system with initial data from a small neighborhood of $\mathcal{A}_{\text{hidden}}$ lead to the localization of the same set $\mathcal{A}_{\text{hidden}}$ (all these trajectories densely trace the set $\mathcal{A}_{\text{hidden}}$). We remark that for the computed trajectories it is observed Zhukovsky instability and the positiveness of Lyapunov exponent [17, 18][5].

[5] Lyapunov exponents (LEs) were introduced by Lyapunov for the analysis of stability by the first approximation for *regular* time-varying linearizations, where negativeness of the largest Lyapunov exponent indicated stability. Later Chetaev proved that for *regular* time-varying linearizations positive Lyapunov exponent indicated instability (a gap in his work is discussed and filled in [18]). While there is no general methods for checking regularity of linearization and there are known Perron effects [18] of the largest Lyapunov exponent sign inversions for non regular time-varying linearizations, computation of Lyapunov exponents for linearization of nonlinear autonomous system along non stationary trajectories is widely used for investigation of chaos, where positiveness of the largest Lyapunov exponent is often considered as indication of chaotic behavior in considered nonlinear system.



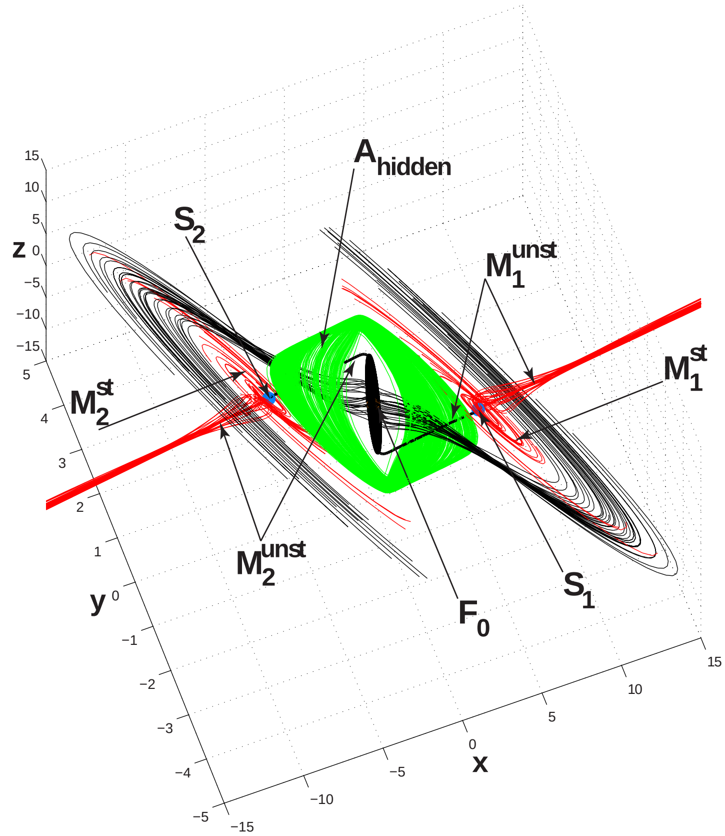




Figure 1: Equilibrium, stable manifolds of saddles, and localization of hidden attractor.

By the above and with provision for the remark on the existence, in system, of locally stable zero equilibrium and two saddle equilibria, we arrive at the conclusion that in $\mathcal{A}_{\text{hidden}}$ a hidden strange attractor is computed.

We study now a behavior of the system in a neighborhood of equilibria. The considered system has three stationary points: the stable zero point $F_0$ and the symmetric saddles $S_1$ and $S_2$. To zero equilibrium $F_0$ correspond the eigenvalues $\lambda_1^{F_0} = -7.9591$ and $\lambda_{2,3}^{F_0} = -0.0038 \pm 3.2495i$ and to the saddles $S_1$ and $S_2$ correspond the eigenvalues $\lambda_1^{S_{1,2}} = 2.2189$ and $\lambda_{2,3}^{S_{1,2}} = -0.9915 \pm 2.4066i$. The behavior of trajectories of system in a neighborhood of equilibria is shown in Fig. 1. Here $M_{1,2}^{\text{unst}}$ are unstable manifolds, corresponding to the eigenvalues $\lambda_1^{S_{1,2}}$, $M_{1,2}^{\text{st}}$ are stable manifolds, corresponding to the eigenvalues $\lambda_{2,3}^{S_{1,2}}$.

It is also can be mentioned here that the existence of hidden attractor $\mathcal{A}_{\text{hidden}}$ is not due to the fact that the nonlinearity is a piecewise constant function. If replace nonlinearity $\text{sat}(\sigma)$ by smooth nonlinearity $\tanh(\sigma)$ then hidden attractor can also be found.

# 4 Conclusions

In the present Letter the application of special analytical-numerical algorithm for hidden attractor localization is discussed and the existence of such hidden attractor in Chua's circuits is demonstrated.

# References


[1] Lorenz E.N. Deterministic nonperiodic flow // J. Atmos. Sci. 1963. V.20. 130-141

[2] Rossler O.E. An Equation for Continuous Chaos // Physics Letters. 1976. V.57A. N5. 397-398

[3] Chua L.O., Lin G.N. Canonical Realization of Chua's Circuit Family // IEEE Transactions on Circuits and Systems. 1990. V.37. N4. 885-902

[4] Chen, G. & Ueta, T. Yet another chaotic attractor, // Int. J. Bifurcation and Chaos. 1999. N9, 1465–1466.

[5] Leonov G.A. On harmonic linearization method.
Doklady Akademii Nauk. Physcis. 2009. 424(4), pp. 462–464.

[6] Leonov G.A. On harmonic linearization method
Automation and remote controle. 2009. 5, pp. 65–75.

[7] Leonov G.A. On Aizerman problem // Automation and remote control. 2009. N7, pp. 37–49.

[8] Leonov G.A. Effective methods for periodic oscillations search in dynamical systems // App. math. & mech. 2010. 74(1), pp. 37–73.

[9] Leonov G.A., Vagaitsev V.I., Kuznetsov N.V. Algorithm for localizing Chua attractors based on the harmonic linearization method // Doklady Mathematics. 2010. 82(1), 663–666.

[10] Leonov G.A., Bragin V.O., Kuznetsov N.V. Algorithm for Constructing Counterexamples to the Kalman Problem // Doklady Mathematics. 2010. 82(1), pp. 540–542.

[11] Chua L.O. A Zoo of Strange Attractors from the Canonical Chua's Circuits // Proceedings of the IEEE 35th Midwest Symposium on Circuits and Systems (Cat. No.92CH3099-9). 1992. V.2. 916-926

[12] Chua L.O. A Glimpse of Nonlinear Phenomena from Chua's Oscillator // Philosophical Transactions: Physical Sciences and Engineering. 1995. V.353. N1701. 3-12

[13] Luo A.C. J., Xue B. An analytical prediction of periodic flows in the chua circuit system // Journal of Bifurcation and Chaos, Vol. 19, No. 7, 2009, pp. 2165-2180.

[14] Bilotta E., Pantano P. A gallery of Chua attractors // World scientific series on nonlinear science, Series A. 2008. V.61

[15] Shi Z., Hong S., and Chen K. Experimental study on tracking the state of analog Chua's circuit with particle filter for chaos synchronization // Physics Letters A, Vol. 372, Iss. 34, 2008, pp. 5575–5580

[16] Anishchenko V.S., Kapitaniak T., Safonova M.A. and Sosnovzeva O.V. Birth of double-double scroll attractor in coupled Chua circuits // Physics Letters A, Vol. 192, Iss. 2-4, 1994, pp. 207–214

[17] Leonov G.A. Strange attractors and classical stability theory. St.Petersburg university Press. 2008.

[18] Leonov G.A., Kuznetsov N.V., Time-Varying Linearization and the Perron effects, International Journal of Bifurcation and Chaos, Vol. 17, No. 4, 2007, pp. 1079-1107.