Computation of hidden oscillations in dynamical systems

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Joint research & educational project within the framework of agreement between University of Jyväskylä and St. Petersburg State Univ.

Prof. G.A.Leonov

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- 4 Master students, PhD students
- 2 defended PhDs (2008, 2009) + 2 next PhD defences (Nov

Outline

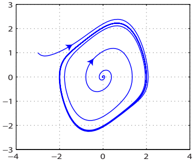
- Self-exciting oscillations and computations
- Hidden oscillations
- Hidden oscillations: engineering applications
- Hidden oscillations: fundamental problems
  - hidden oscillation in 2d space: limit cycles in 16th Hilbert problem
  - hidden oscillation in 3d space: Aizerman's and Kalman's conjectures
  - chaotic hidden oscillation: hidden strange attractors
- Hidden oscillation localization: analytical-numerical approach
- Examples: counterexamples Aizerman's and Kalman's conjectures
- Examples: First chaotic hidden attractor in Chua's circuit

Classical approach to computation of oscillations

localization of self-exciting oscillations and attractors -
standard computation: 1) determine equilibria 2) after transient process trajectory, starting from a point of unstable manifold in neighborhood of unstable equilibrium, reaches an oscillation and identifies it.

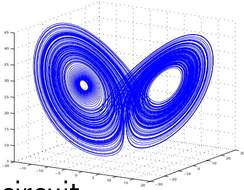
Van der Pol

$$\dot{x} = y$$
$$\dot{y} = -x + \varepsilon(1 - x^2)y$$



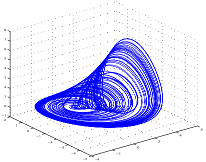
Lorenz system

$$\dot{x} = -\sigma(x - y)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$



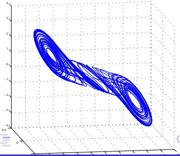
Rössler system

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = bx + z(x - c)$$



Chua's circuit

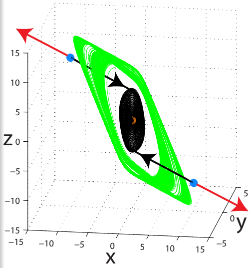
$$\dot{x} = \alpha(y - x + f(x))$$
$$\dot{y} = x - y + z$$
$$\dot{z} = -\beta y - \gamma z$$
$$f(x) = bx + (a\text{-}b)\text{sat}(x)$$



Hidden oscillations: how to compute?

hidden oscillations and hidden attractors — basin of attraction does not contain neighborhoods of equilibria
[Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Localization of hidden Chua's attractors. *Phys. Lett. A*, 2011]

✓standard computation (*trajectory from a neighborhood of unstable equilibrium reaches and identifies attractor*) does not work (all equilibria are stable or not in the basin).



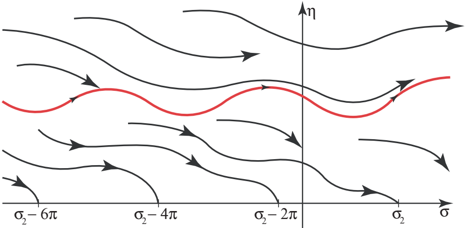
✓ integration with random initial data is unlikely to furnish the desired result, since a basin of attraction can be very small and system's dimension can be large

The problem is: How to step away from equilibria and choose initial data in the attraction domain of hidden oscillation?

Hidden oscillations: examples in engineering applications

Gubar' N.A. (1961),
J. Appl. Math. Mech, 25,
pp. 1519–1535.

$$\dot{\eta} = \alpha\eta - (1 - a\alpha)(\operatorname{sign}\sin(\sigma) - \gamma),$$
$$\dot{\sigma} = \eta - a(\operatorname{sign}\sin(\sigma) - \gamma)$$

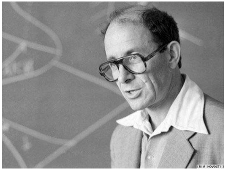


Hidden oscillations in Phase-locked loops (PLL)

Lauvdal T., Murray R.M., Fossen T.I., Stabilization of Integrator Chains in the Presence of Magnitude and Rate Saturations; a Gain Scheduling Approach, *Proceeding of CDC, 1997*:
"Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22 [Boeing]*) stronger theoretical understanding is required"*

Hidden oscillations in aircrafts

Hidden oscillations (2d): limit cycles in 16th Hilbert problem



- D.Hilbert: Limit cycles (LC) number & disposition in two-dimensional polynomial system
  $\dot{x} = P_n(x, y), \dot{y} = Q_n(x, y) = ax + by + cx^2 + dxy + ey^2 ...$
- A.Kolmogorov: Calculation of limit cycles in two-dimensional quadratic systems
- V.Arnold: Estimation of parameters domain corresponding to existence of limit cycles

V. Arnold wrote (2005): *To estimate the number of LCs of square vector fields on plane, A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.&Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!...The fact that this did not occur suggests that the above-mentioned domains are, apparently, small.*

hidden oscillations — nested limit cycles

Hidden oscillations in 2d: analytical approach

$$\dot{x} = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y$$

- N.N. Bautin 1949-1952: 3 limit cycles [around one focus]
- I.G.Petrovskii, E.M.Landis 1955–1959: **only** 3 limit cycles
- L.Chen & M.Wang, S.Shi 1979-80: 4 limit cycles [(1,3), 2 focuses]
- R. Bamon 1985: number of LC in QS is finite

Number of limit cycles $H(n)$: $H(2) \geqslant 4$

- small-amplitude limit cycles: analytical methods
  - Lyapunov quantity (focus value, Poincare-Lyapunov constant) weak focus & Andronov-Hopf bifurcation
- normal-amplitude limit cycles: analytical & numerical methods

numerical methods: hidden oscillation — nested and semistable cycles

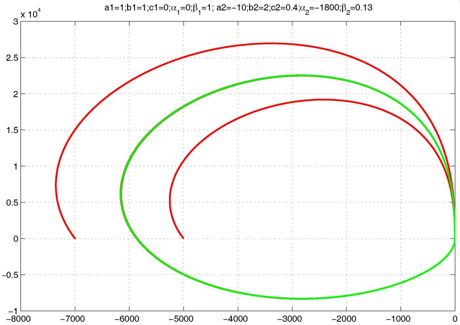
Visualization of 4 normal size limit cycles in QS

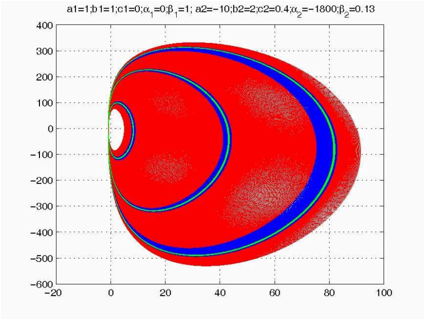
$$\begin{aligned} \dot{x} &= x^2 + xy + y \\ \dot{y} &= ax^2 + bxy + cy^2 + \alpha x + \beta y \end{aligned} \qquad (1)$$

$c \in (1/3, 1),\ \alpha = -\varepsilon^{-1},\ bc < 1,\ b > a + c,\ 2c < b + 1,\ 4a(c-1) > (b-1)^2,\ \beta = 0$

Theorem. *For sufficiently small ε system (1) has three limit cycles: one to the left of line $\{x = -1\}$ and two to the right of it.*

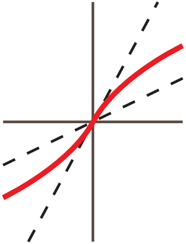
Increase β and get four normal size limit cycles.





Hidden oscillations (3d): Aizerman's & Kalman's conjectures

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\forall \mathbf{z}(t, \mathbf{z}_0) \to 0$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma$, $\varphi' < k_2$, $\forall \mathbf{x}(t, \mathbf{x}_0) \to 0$?



$1949: \; k_1 < \varphi(\sigma)/\sigma < k_2$ $\qquad$ $1957: \; k_1 < \varphi'(\sigma) < k_2$

In general, conjectures are not true (Aizerman's $n \geq 2$, Kalman's $n \geq 4$).
Nonlinearity can be in linear stability sector but periodic solutions exist.

✓ Bragin, Leonov, Kuznetsov, Vagaitsev (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int., V.50, N4, 511-544 (survey)*

Kalman problem (Kalman's conjecture, 1957)

if $\dot{z}=(A+kbc^*)z$, is asympt. stable $\forall\, k\in(k_1,k_2)$: $\forall z(t,z_0)\to 0$
$\dot{x} = Ax + b\varphi(\sigma),\ \sigma = c^*x, \varphi(0) = 0,\ k_1 < \varphi'(\sigma) < k_2,\ \forall x(t,x_0)\to 0?$

- Fitts R., 1966: series of counterexamples in $\mathbb{R}^4$, nonlinearity $\varphi(\sigma) = \sigma^3$ (some of them were reported being false, but some — true)
- Barabanov N., 1979–1988:
  - proof Kalman's conj. is true for $\mathbb{R}^3$
  - analytical method for "counterexamples" construction in $\mathbb{R}^4$ with linear stability sector $(0, k_2)$ and nonlinearity "close to" $\mathrm{sign}(\sigma)$

  later some "gaps" were reported by Glutsyuk, Meisters, Bernat & Llibre
- Leonov G., 1996: proof Kalman's conj. is true for $\mathbb{R}^3$ (by freq. methods)
- Bernat J. & Llibre J., 1996: analytical-numerical method for "counterex." construction in $\mathbb{R}^4$ with lin. stab. sector $(0, k_2)$ and nonl. "close to" $\mathrm{sat}(\sigma)$
- Leonov G., Kuznetsov N., Bragin V., 2010:
  - analytical-numerical method for counterexamples construction
  - counterexample system in $\mathbb{R}^4$, nonl. $\tanh(\sigma)$: $0 < \tanh'(\sigma) \leqslant 1$

Describing function method (DFM) can lead to untrue results:
no periodic solution for Aizerman's or Kalman's conditions by DFM

$(1)\ \dot{\mathbf{x}}=\mathbf{P}_0\mathbf{x}+\varphi(\mathbf{x})$ $(2)\ \dot{\mathbf{x}}=\mathbf{P}_0\mathbf{x}+\varepsilon\boldsymbol{\varphi}(\mathbf{x})$ $(3)\ \dot{\mathbf{x}}^j=\mathbf{P}_0\mathbf{x}^j+\varepsilon_j\boldsymbol{\varphi}(\mathbf{x}^j)$

small ε allows one to justify math. strictly DFM for (2) & to determine a stable nontrivial periodic solution $\mathbf{x}^0(t)$ — *oscillating attractor* $\mathcal{A}_0$.

Localization of attractor $\mathcal{A}$ in (1): numerically follow transformation of $\mathcal{A}_j$ with increasing $j=0,..,m$ ($\varepsilon_m=1, \mathcal{A}_m=\mathcal{A}$) [special continuation method]

1. if all points of $\mathcal{A}_0$ are in the attraction domain of $\mathcal{A}_1$ (oscillating attractor of (3) with $j=1$), then solution $\mathbf{x}^1(t)$ can be determined numerically by starting a trajectory of (3) with $j=1$ from initial point $\mathbf{x}^0(0)$. If in computational process $\mathbf{x}^1(t)$ is not fallen to equilibria and is not $\to\infty$ (on suff. large $[0,T]$), then $\mathbf{x}^1(t)$ computes attractor $\mathcal{A}_1$. Then perform similar procedure for (3) with $j=2$: by starting trajectory $\mathbf{x}^2(t)$ of (3) with $j=2$ from init. point $\mathbf{x}^1(T)$ (last point on previous step) we compute $\mathcal{A}_2$. And so on.

2. in the change from system (2) to (3) with $j=1$, it's observed loss of stability bifurcation and vanishing of attractor $\mathcal{A}_0$(or $\mathcal{A}_{j-1}$ on j-th step).

Harmonic Balance & Describing Function Method

Why it is necessary to justify Harmonic balance and DFM?

$$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}), \ \ \psi(0) = 0 \quad (1)$$

$$W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q}$$

$$\mathrm{Im} W(i\omega) = 0$$

$$k = -(\mathrm{Re} W(i\omega))^{-1}$$

$$\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi(\mathbf{r}^*\mathbf{x})$$

$$\mathbf{P_0} = \mathbf{P} + k\mathbf{q}\mathbf{r}^*$$

$$\varphi(\sigma) = \psi(\sigma) - k\sigma$$

$$\mathbf{P}_0 : \lambda_{1,2} = \pm i\omega, \ \ \mathrm{Re}\lambda_{j>2} < 0$$

DFM: exists periodic solution $\sigma(t) = \mathbf{r}^*\mathbf{x}(t) \approx a\cos\omega t$

$$a : \ \int_0^{2\pi/\omega} \psi(a\cos\omega t)\cos\omega t \mathrm{d}t = ka\int_0^{2\pi/\omega}(\cos\omega t)^2 \mathrm{d}t$$

Aizerman: if $\dot{\mathbf{z}} = (\mathbf{A} + \mu\mathbf{b}\mathbf{c}^*)\mathbf{z}$, is asympt. stable $\forall\, \mu \in (\mu_1, \mu_2)$ then $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $\mu_1 < \varphi(\sigma)/\sigma < \mu_2$, all $\mathbf{x}(t, \mathbf{x}_0) \to 0$

DFM: since (1) is stable in sector $\mu_1 < \varphi(\sigma)/\sigma < \mu_2$ due to Aizerman

$\Rightarrow$ k from DFM $: k < \mu_1, \mu_2 < k \Rightarrow k\sigma^2 < \psi(\sigma)\sigma, \psi(\sigma)\sigma < k\sigma^2$

$\Rightarrow \forall a \neq 0 : \ \int_0^{2\pi/\omega}(\psi(a\cos\omega t)\, a\cos\omega t - k(a\,\cos\omega t)^2)\mathrm{d}t \neq 0$

$\Rightarrow$ no periodic solutions by DFM, but counterexamples are well known

Periodic solution by justified DFM (scalar case)

$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\psi(\mathbf{r}^*\mathbf{x}) : W(p) = \mathbf{r}^*(\mathbf{P} - p\mathbf{I})^{-1}\mathbf{q}, \operatorname{Im} W(i\omega) = 0, k = -(\operatorname{Re} W(i\omega))^{-1}$

$\dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\boldsymbol{\varphi}(\mathbf{r}^*\mathbf{x}) : \mathbf{P}_0 = \mathbf{P} + \mathbf{k}\mathbf{q}\mathbf{r}^*, \lambda_{1,2}^{\mathbf{P}_0} = \pm i\omega, \operatorname{Re}\lambda_{j>2}^{\mathbf{P}_0} < 0, \varphi = \psi - k\mathbf{r}^*\mathbf{x}$

$\mathbf{x} = \mathbf{S}\mathbf{y}, \mathbf{A} = \mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}, \mathbf{b} = (b_1, b_2, \mathbf{b}_3)^* = \mathbf{S}^{-1}\mathbf{q}, \mathbf{c}^* = (1, 0, \mathbf{c}_3)^* = \mathbf{r}^*\mathbf{S}$

Harmonic linearization, linear transformation, small parameter method

$$\dot{y}_1 = -\omega y_2 + b_1\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \qquad y_1(t) = \cos(\omega t)y_1(0) + O(\varepsilon)$$
$$\dot{y}_2 = \omega y_1 + b_2\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \qquad y_2(t) = \sin(\omega t)y_1(0) + O(\varepsilon)$$
$$\dot{\mathbf{y}}_3 = \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}_3\varepsilon\varphi(y_1 + \mathbf{c}_3^*\mathbf{y}_3) \qquad \mathbf{y}_3(t) = e^{\mathbf{A}_3 t}\mathbf{y}_3(0) + \mathbf{O_{n-2}}(\varepsilon)$$
$$\mathbf{y}_3^*(\mathbf{A}_3 + \mathbf{A}_3^*)\mathbf{y}_3 \le -2d|\mathbf{y}_3|^2 \qquad t \in (0, T]$$

$$\mathbf{y}(0) \in \Omega = \{y_1 \in [a_1, a_2], y_2 = 0, |\mathbf{y}_3| \le D\varepsilon\}, \mathbf{F}\mathbf{y}(0) = \mathbf{y}(T), T = \frac{2\pi}{\omega} + O(\varepsilon)$$

Theorem. If exists $a_0 > 0 : \ \Phi(a_0) = \int\limits_0^{2\pi/\omega} \varphi\big(\cos(\omega t)a_0\big)\cos(\omega t)dt = 0$ and $b_1 \frac{d\Phi(a)}{da}\Big|_{a=a_0} < 0$ then exists periodic solution $\mathbf{x}(t) = \mathbf{S}\mathbf{y}(t)$ with initial data $\mathbf{x}(0) = \mathbf{S}(a_0 + O(\varepsilon), 0, \mathbf{O_{n-2}}(\varepsilon))^*$.
For all solutions with initial data suff. close to $\mathbf{x}(0)$ the modulus of their difference with $\mathbf{x}(t)$ is uniformly bounded $\forall t > 0$).

Justification of DFM (critical case)

$\dot{\mathbf{x}} = \mathbf{P}\mathbf{x} + \mathbf{q}\varphi_\varepsilon(r^*\mathbf{x}),\ \ \mathbf{x} \in \mathbb{R}^n$

Eigs($\mathbf{P}$): $\lambda_{1,2} \pm i\omega_0$,

$\lambda_{j>2}$(Re $\lambda_{j>2} < 0$)

$\Rightarrow \exists \mathbf{x} = Sy$:

$$\dot{y}_1 = -\omega_0 y_1 + b_1\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$

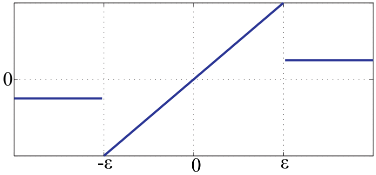
$$\dot{y}_2 = +\omega_0 y_1 + b_2\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$

$$\dot{\mathbf{y}}_3 = \mathbf{A}_3\mathbf{y}_3 + \mathbf{b}\varphi_\varepsilon(y_1 + \mathbf{c}_3^*\mathbf{y}_3)$$

$\mathbf{A}_3$–stable $(n-2)\times(n-2)$-matrix

$\mathbf{b}, \mathbf{c}$ — $(n-2)$-vectors.

$$\varphi_\varepsilon(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon \\ \mathrm{sign}(\sigma)M\varepsilon^3, & \forall|\sigma| > \varepsilon \end{cases}$$



Stab. sector: $\mu\sigma \geqslant \varphi_\varepsilon \geqslant 0$

Theorem. If $b_1 < 0$ and $0 < (\mu b_2\omega_0(\mathbf{c}_3^*\mathbf{b}_3 + b_1) + b_1\omega_0^2)$ then for suff. small ε exists a stable periodic solution with the initial data

$$y_1(t) = -\sin(\omega_0 t)y_2(0) + O(\varepsilon),\ y_2(t) = \cos(\omega_0 t)y_2(0) + O(\varepsilon),\ \mathbf{y}_3(t) = O(\varepsilon)$$

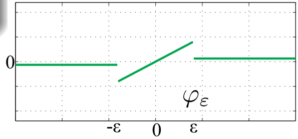
$$y_1(0) = O(\varepsilon^2),\ y_2(0) = -\sqrt{\frac{\mu(\mu b_2\omega_0(\mathbf{c}_3^*\mathbf{b}_3 + b_1) + b_1\omega_0^2)}{-3\omega_0^2 M b_1}} + O(\varepsilon),\ \mathbf{y}_3(0) = O(\varepsilon^2)$$

Analytical-numerical algorithm: periodic solution localization

Thm. $\mathbf{P}_0 = \mathbf{P} - k\mathbf{q}r^*$: $\lambda_{1,2} = \pm i\omega_0, \mathrm{Re}\,\lambda_{j>2} < 0$, for small $\varepsilon\, \exists$ periodic sol: $\mathbf{x}_0 = Sy(0), \mathbf{x}^\varepsilon(0,\mathbf{x}_0) = \mathbf{x}^\varepsilon(T,\mathbf{x}_0)$

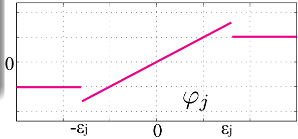
$$(1)\ \dot{\mathbf{x}}^\varepsilon = \mathbf{P}_0\mathbf{x}^\varepsilon + \mathbf{q}\varphi_\varepsilon(r^*\mathbf{x}^\varepsilon)$$

$$\varphi_\varepsilon(\sigma) = \begin{cases} \mu\sigma, & \forall|\sigma| \le \varepsilon \\ \mathrm{sign}(\sigma)M\varepsilon^3, & \forall|\sigma| > \varepsilon \end{cases}$$



Multistep numerical proc.: $\mathbf{x}^1(0) = Sy(0)$ by Thm
ε_1 small $\approx \varepsilon, \varphi_1(\sigma) \approx \varphi_\varepsilon(\sigma), \mathbf{x}^1(\mathbf{x}_0, t) \approx \mathbf{x}^\varepsilon(\mathbf{x}_0, t)$
$\varepsilon_j = (j/m)\sqrt{\mu/M},\ \mathbf{x}^{j+1}(0) = \mathbf{x}^j(T)$

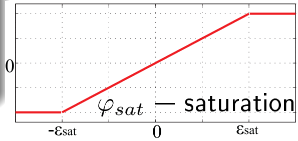
$$(2)\ \ \dot{\mathbf{x}}^j = \mathbf{P}_0\mathbf{x}_j + \mathbf{q}\varphi_j(r^*\mathbf{x}^j),\ \text{step } j = 1, ..., m-1$$



Counterexample to Aizerman's conj: $\exists$ periodic solution in system (3) with continuous $\varphi(\sigma)$ from stable sector from the sector of stability

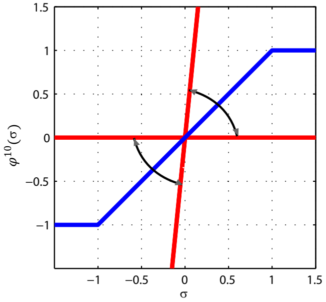
$$(3)\ \dot{\mathbf{x}} = \mathbf{P}_0\mathbf{x} + \mathbf{q}\varphi_{sat}(r^*\mathbf{x}), \quad j = m$$

$$\varepsilon_m = \varepsilon_{sat} = \sqrt{\mu/M}, \varphi_m(\sigma) = \varphi_{sat}(\sigma)$$



Counterexample to Aizerman's and Kalman's conjecture

$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$



Thm: $\varphi(\sigma) = \varphi^0(\sigma)$ $\exists$ periodic solution with $x_1(0) = x_3(0) = x_4(0) = 0,\ x_2(0) = -1.7513$

Aizerman's conjecture: $0 \leqslant \varphi^j(\sigma) \leqslant 1$,

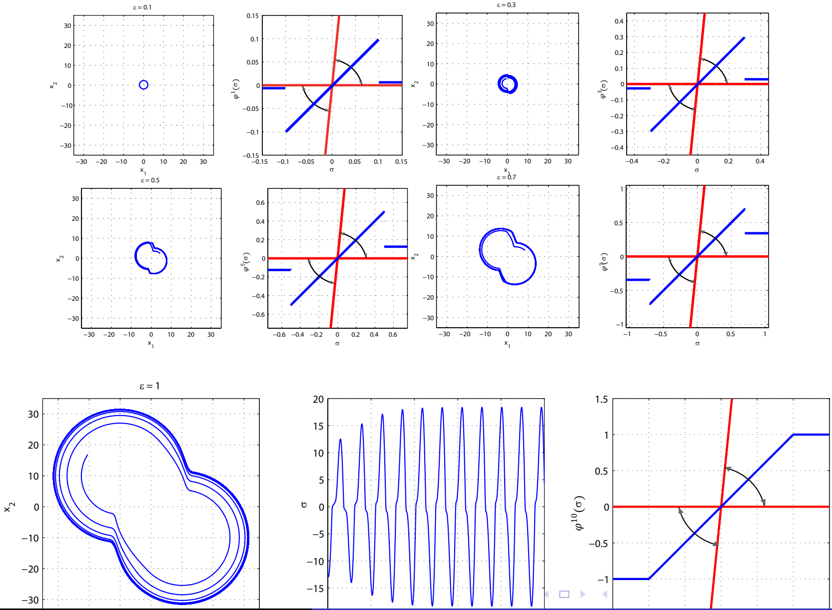
$$\varphi^j(\sigma) = \begin{cases} \sigma, & |\sigma| \leq \varepsilon_j; \\ \operatorname{sign}(\sigma)\varepsilon_j^3, & |\sigma| > \varepsilon_j \end{cases} \quad \varepsilon_j = 0.1, ..., 1, \quad \varphi^{10}(\sigma) = \operatorname{sat}(\sigma)$$

Kalman's conjecture: $iN \leqslant \psi^{i\prime}(\sigma) \leqslant 1 \qquad 0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1$

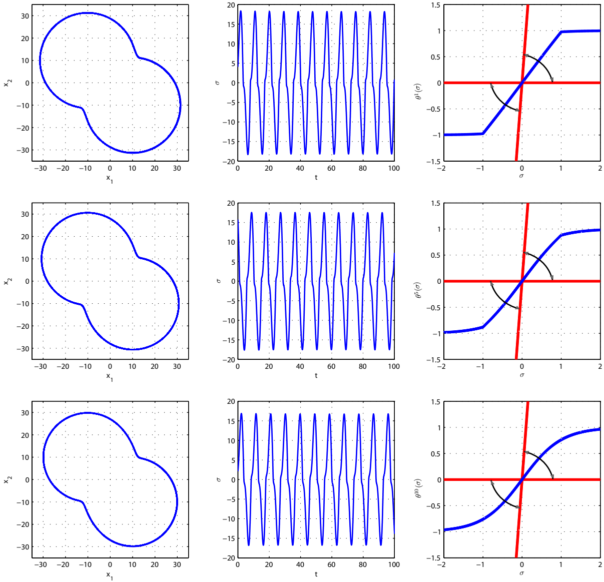
$$\psi^i(\sigma) = \begin{cases} \sigma, & |\sigma| \leq 1; \\ \operatorname{sign}(\sigma) + i(\sigma - \operatorname{sign}(\sigma))N, & |\sigma| > 1 \end{cases} \quad N = 0.01, i = 1, .., 5$$

$$\theta^i(\sigma) = \operatorname{sat}(\sigma) + i(\tanh(\sigma) - \operatorname{sat}(\sigma))/10 \quad i = 1, .., 10 \quad \theta^{10}(\sigma) = \tanh(\sigma)$$

Counterexample to Aizerman problem



Smooth counterexample to Kalman problem



$$\dot{x}_1 = -x_2 - 10\,\varphi(\sigma)$$
$$\dot{x}_2 = x_1 - 10.1\,\varphi(\sigma)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -x_3 - x_4 + \varphi(\sigma)$$
$$\sigma = x_1 - 10.1\,x_3 - 0.1\,x_4$$

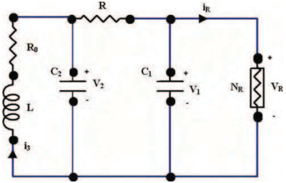
$$\varphi(\sigma) = \theta^i(\sigma) = \mathrm{sat}(\sigma) + i(\tanh(\sigma) - \mathrm{sat}(\sigma))/10$$
$$i = 1, .., 10$$
$$\tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}$$

Counterexample to Kalman problem (i=10) $(0 < \frac{d}{d\sigma}\tanh(\sigma) \leq 1)$ periodic solution exists, linear systems are stable

Attractors in Chua's circuits

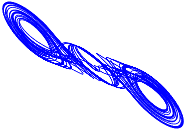
L.Chua (1983)

$$\dot{x} = \alpha\big(y - x - f(x)\big),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$



$$f(x) = m_1 x + \mathrm{sat}(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x+1| + |x-1|)$$

Chua circuit is used in chaotic communications



1983–now: computations of Chua self-excited attractors by standard procedure: trajectory from a neighborhood of unstable equilibrium reaches and identifies attractor. [*Bilotta&Pantano, A gallery of Chua attractors, WorldSci. 2008*]

Could an attractor exists and how lo localize it, if equilibrium is stable?

L.Chua, 1992: If zero equilibrium is stable ⇒ there is no attractor

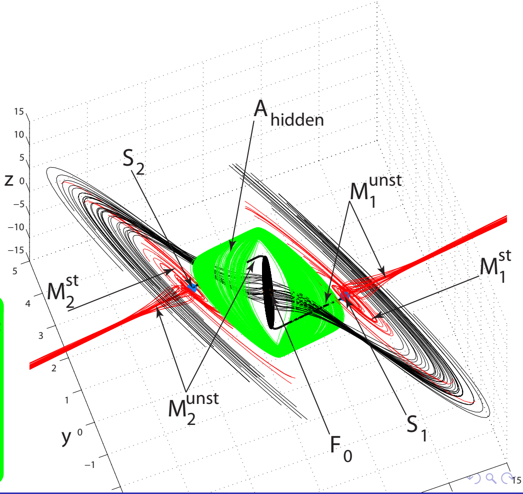
Hidden attractor in classical Chua's system

In 2010 the notion of *hidden attractor* was introduced
and hidden chaotic attractor was found for the first time by the authors
[Leonov G.A., Kuznetsov N.V., Vagaitsev V.I., Physics Letters A, 375(23), 2011]

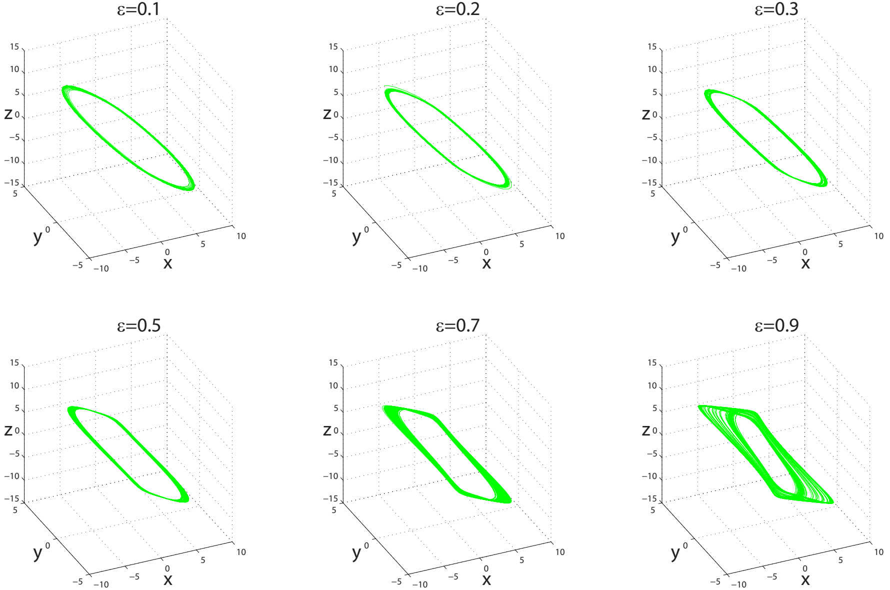
$$\dot{x} = \alpha(y - x - m_1 x - \psi(x))$$
$$\dot{y} = x - y + z, \dot{z} = -(\beta y + \gamma z)$$
$$\psi(x) = (m_0 - m_1)\mathrm{sat}(x)$$

$\alpha = 8.4562, \beta = 12.0732$
$\gamma = 0.0052$
$m_0 = -0.1768, m_1 = -1.1468$

equilibria: stable zero F_0 & 2 saddles $S_{1,2}$
trajectories: 'from' $S_{1,2}$ tend (black) to zero F_0 or tend (red) to infinity;
Hidden chaotic attractor (in green) with positive Lyapunov exponent



Road to chaos: Chua Hidden attractor localization



$$\begin{cases} \dot{x} = F(x), \; x \in \mathbb{R}^n, \quad F(x_0) = 0 \\ x(t) \equiv x_0, A = \frac{dF(x)}{dx}\Big|_{x=x_0} \end{cases} \quad \begin{cases} \dot{y} = Ay + o(y) \\ y(t) \equiv 0, (y = x - x_0) \end{cases} \quad \begin{cases} \dot{z} = Az \\ z(t) \equiv 0 \end{cases}$$

✓stationary: $z(t) = 0$ is exp. stable $\Rightarrow$ $y(t) = 0$ is asympt. stable

$$\begin{cases} \dot{x} = F(x), \quad \dot{x}(t) = F(x(t)) \not\equiv 0 \\ x(t) \not\equiv x_0, A(t) = \frac{dF(x)}{dx}\Big|_{x=x(t)} \end{cases} \quad \begin{cases} \dot{y} = A(t)y + o(y) \\ y(t) \equiv 0, (y = x - x(t)) \end{cases} \begin{cases} \dot{z} = A(t)z \\ z(t) \equiv 0 \end{cases}$$

? nonstationary: $z(t) = 0$ is exp. stable $\Rightarrow ?$ $y(t) = 0$ is asympt. stable

! Perron effect: $z(t) = 0$ is exp. stable(unst), $y(t) = 0$ is exp. unstable(st)

Positive largest Lyapunov exponent doesn't, in general, indicate chaos

Survey: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, International Journal of Bifurcation and Chaos, Vol. 17, No. 4, 2007, pp. 1079-1107 (doi:10.1142/S0218127407017732)

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Questions and remarks