IFAC Proceedings Volumes (IFAC-PapersOnline) 3, PART 1, 2007, pp. 49-52 doi:10.3182/20070829-3-RU-4912.00007

Synchronization of two metronomes

Kuznetsov N.V., Leonov G.A., Nijmeijer H., Pogromsky A.

Draft [1]

Abstract. The in-phase synchronization of two metronomes placed on a freely moving base is examined. The existence of an inphase regime is proved under special conditions.

Keywords: in-phase synchronization of two metronomes, pendulums, in-phase regime, anti-phase synchronization, Huygens

1 Introduction. Anti-phase and in-phase synchronization in Huygens problem

The synchronization phenomenon of two pendulum clocks was first discovered by Cristian Huygens in 1662. He observed what is now called the anti-phase synchronization of two pendula of the clocks attached to a common support beam. Regardless of the initial conditions those two pendula converged after some transient to an oscillatory regime characterized by identical frequency of the oscillations, while the two pendula angles moved in anti phase. Huygens [Huygens 1669,1986] found an explanation of this phenomenon noticing that imperfect synchronization resulted in small beam oscillation that in turn drove the pendula towards the agreement. Though his explanation is physically correct, rigorous analytical results become available later on with invention of differential calculus. 300 years of Huygens' discovery it turned out that this phenomenon finds a lot of potential applications in different fields of science and engineering. For some related analytical results, see e.g.[Bennet *et. al.,* 2005, Pogromsky *et. al.,* 2005].

Together with anti-phase oscillations, a similar setup with two metronomes on a common support demonstrates also in-phase synchronization, where metronomes' pendula agree not only in frequency but also in angles [Oud *et. al.,* 2006].

In the book [Blekhman 1988], Blekhman also discusses Huygens' observations, and recounts the results of a laboratory reproduction of the coupled clocks as well as presenting a theoretical analysis of oscillators coupled through a common supporting frame. He predicted that both in-phase and anti-phase motions are stable under the same circumstances.

The problem of analytical study of in-phase synchronization turns out to be more difficult. The present paper addresses this problem for the model of two metronomes on the common support proposed in [Pantaleone, 2002].

2 Problem statement: in-phase synchronization

We consider a system consisting of two metronomes resting on a light wooden board that sits on two empty soda cans. The motion of such system can be described [Pantaleone, 2002] by the following equation

$$\begin{aligned} & m(l\ddot{\phi}_i + \ddot{x}\cos\phi_i) + mg\sin\phi_i = \\ & = \kappa^{esc}\big(1 - \frac{\widehat{\phi_i}^{\,2}}{\Phi^2}\big)\dot{\phi}_i = f_{esc}(\phi_i)\dot{\phi}_i \\ & M\ddot{x} + m\sum_{i=1}^{2}\big(l\ddot{\phi}_i\cos\phi_i - l\dot{\phi}_i^{\,2}\sin\phi_i + \ddot{x}\big) = 0 \end{aligned} \tag{1}$$

[1]Nikolay V. Kuznetsov, nkuznetsov239 at gmail.com (correspondence author)

Here m is a mass of each weight of metronomes, M is a mass of platform, ϕ_i is an angle of deviation of the i-th pendulum of metronome from a vertical, l is a length of the pendulum of metronome, $f_{esc}(\cdot)$ is an internal force of metronome, (κ^{esc} is a small parameter, $\widehat{\phi_i} = \phi_i \mod 2\pi$), g is a gravitational acceleration, and x is a horizontal displacement of platform beginning from equilibrium.

We find out conditions under which the inphase regime occurs.

For the sequel we need in new variables

$$\theta_+ = \frac{\phi_1 + \phi_2}{2}, \quad \theta_- = \frac{\phi_1 - \phi_2}{2}.$$

In this case by trigonometric formulas we obtain

$$\begin{aligned} \frac{\sin\phi_1 + \sin\phi_2}{2} &= \sin\theta_+ \cos\theta_-, \\ \frac{\sin\phi_1 - \sin\phi_2}{2} &= \sin\theta_- \cos\theta_+, \\ \frac{\cos\phi_1 + \cos\phi_2}{2} &= \cos\theta_+ \cos\theta_-, \\ \frac{\cos\phi_1 - \cos\phi_2}{2} &= \sin\theta_- \sin\theta_+. \end{aligned}$$

The second equation of (1) gives the following expression for the acceleration $\ddot{x}$ of the platform

$$\ddot{x} = -\frac{ml\big((\sin\phi_1)'' + (\sin\phi_2)''\big)}{(M+2m)}.$$

This implies that, written a half-sum and half-difference of equations for the motion of weights of metronomes (1) in new variables θ_+, θ_-, we obtain

$$\begin{aligned} & m(l\ddot{\theta}_+ - \frac{2ml(\sin\theta_+\cos\theta_-)''}{M+2m}\cos\theta_+\cos\theta_- + \\ & +g\sin\theta_+\cos\theta_-) = (f_{esc}(\phi_1) + f_{esc}(\phi_2))/2, \\ & m(l\ddot{\theta}_- - \frac{2ml(\sin\theta_+\cos\theta_-)''}{M+2m}\sin\theta_-\sin\theta_+ \\ & +g\sin\theta_-\cos\theta_+) = (f_{esc}(\phi_1) - f_{esc}(\phi_2))/2, \end{aligned} \tag{2}$$

$$\begin{aligned} & f_{esc}(\phi_1)\dot{\phi_1} + f_{esc}(\phi_2)\dot{\phi_2} = \kappa^{esc}\big(1 - \frac{\phi_1^2}{\Phi^2}\big)\dot{\phi_1} + \\ & +\kappa^{esc}\big(1 - \frac{\phi_2^2}{\Phi^2}\big)\dot{\phi_2} = \\ & = \kappa^{esc}\big(2\dot{\theta_+} - \frac{1}{\Phi^2}\big((\theta_+ + \theta_-)^2(\dot{\theta_+} + \dot{\theta_-}) + \\ & +(\theta_+ - \theta_-)^2(\dot{\theta_+} - \dot{\theta_-})\big)\big) = \\ & = \kappa^{esc}\big(2\dot{\theta_+} - \frac{1}{\Phi^2}\big(2\theta_+^2\dot{\theta_+} + 4\theta_+\theta_-\dot{\theta_-} + 2\theta_-^2\dot{\theta_+}\big)\big), \\ & f_{esc}(\phi_1)\dot{\phi_1} - f_{esc}(\phi_2)\dot{\phi_2} = \kappa^{esc}\big(1 - \frac{\phi_1^2}{\Phi^2}\big)\dot{\phi_1} - \\ & -\kappa^{esc}\big(1 - \frac{\phi_2^2}{\Phi^2}\big)\dot{\phi_2} = \\ & = \kappa^{esc}\big(2\dot{\theta_-} - \frac{1}{\Phi^2}\big((\theta_+ + \theta_-)^2(\dot{\theta_+} + \dot{\theta_-}) - \\ & -(\theta_+ - \theta_-)^2(\dot{\theta_+} - \dot{\theta_-})\big)\big) = \\ & = \kappa^{esc}\big(2\dot{\theta_-} - \frac{1}{\Phi^2}\big(2\theta_+^2\dot{\theta_-} + 4\theta_+\theta_-\dot{\theta_+} + 2\theta_-^2\dot{\theta_-}\big)\big). \end{aligned} \tag{3}$$

It follows that in the mechanical system there can occurs an inphase regime ($2\theta_- = \phi_1 - \phi_2 = 0 \Rightarrow \phi_1 \equiv \phi_2$), in which for the half-sum of angles of deviation of metronomes pendulum $\theta = \theta_+$ satisfies the equation

$$\begin{aligned} & m(l\ddot{\theta} - \frac{2ml(\sin\theta)''}{M+2m}\cos\theta + g\sin\theta) = \\ & = \kappa^{esc}\big(1 - \frac{\theta^2}{\Phi^2}\big)\dot{\theta}. \end{aligned} \tag{4}$$

Having performed the transformations

$$\begin{aligned} & m(l\ddot{\theta} - \frac{2ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)}{M+2m}\cos\theta + g\sin\theta) = \\ & = \kappa^{esc}\big(1 - \frac{\theta^2}{\Phi^2}\big)\dot{\theta}, \end{aligned} \tag{5}$$

$$\begin{aligned} & \ddot{\theta}(1 - \frac{2m\cos^2\theta}{M+2m}) + \dot{\theta}^2\frac{2m\sin\theta\cos\theta}{M+2m} + \frac{g}{l}\sin\theta = \\ & = \frac{\kappa^{esc}}{ml}\big(1 - \frac{\theta^2}{\Phi^2}\big)\dot{\theta}, \end{aligned} \tag{6}$$

$$\begin{aligned} & \ddot{\theta}\frac{M+2m\sin^2\theta}{M+2m} + \dot{\theta}^2\frac{m\sin 2\theta}{M+2m} + \frac{g}{l}\sin\theta = \\ & = \frac{\kappa^{esc}}{ml}\big(1 - \frac{\theta^2}{\Phi^2}\big)\dot{\theta}, \end{aligned} \tag{7}$$

for

$$\varepsilon_{m,M}(\theta) = \frac{2m\cos^2\theta}{M+2m\sin^2\theta},$$

$$\kappa^{esc}_{ml} = \frac{\kappa^{esc}}{ml}, \quad \varepsilon_{\frac{2m}{M}} = \frac{2m}{M},$$

we obtain

$$\ddot{\theta} - \dot{\theta}F(\theta,\kappa^{esc}) + \dot{\theta}^2 H(\theta) + G(\theta) = 0, \tag{8}$$

$$\begin{aligned} & F(\theta,\kappa^{esc}) = \kappa^{esc}\frac{M+2m}{ml(M+2m\sin^2\theta)}\big(1 - \frac{\theta^2}{\Phi^2}\big) = \\ & = \kappa^{esc}_{ml}\big(1 - \frac{\theta^2}{\Phi^2}\big)(1 + \varepsilon_{m,M}(\theta)), \\ & H(\theta) = \frac{m\sin 2\theta}{M+2m\sin^2\theta} = \tan\theta\varepsilon_{m,M}(\theta), \\ & G(\theta) = \frac{g(M+2m)\sin\theta}{l(M+2m\sin^2\theta)} = \frac{g\sin\theta}{l}(1 + \varepsilon_{m,M}(\theta)) \end{aligned}$$

Then for linearization at the equilibrium $\theta = 0, \dot{\theta} = 0$ we have

$$\begin{aligned} & \dot{\theta} = \eta \\ & \dot{\eta} = -G'_\theta(\theta)|_{\theta=0,\eta=0}\theta + F(\theta,\kappa^{esc})|_{\theta=0,\eta=0}\eta = \\ & = -\frac{g(M+2m)}{Ml}\theta + \frac{\kappa^{esc}(M+2m)}{Mml}\eta \end{aligned}$$

Since $\frac{\kappa^{esc}(M+2m)}{Mml} > 0$, this implies the instability of the zero solution and the unwinding of the phase trajectory for small θ.

Following the strategy in [Pogromsky *et. al.*, 2005; Bennet *et. al.*, 2005], we consider the question of the existence and setting of the inphase regime.

3 Proof of the existence of a in-phase periodic regime

Having performed some transformation in system (4) and taking into account the relations

$$\kappa = \sqrt{g/l}, \quad \varepsilon_M = \frac{m}{M+2m},$$

$$\delta_M(\theta) = \varepsilon_M(\ddot{\theta}_2 \cos^2\theta - \dot{\theta}^2 \sin 2\theta),$$

$$f_{ml}^{esc}(\theta) = \kappa_{ml}^{esc}\big(1 - \frac{\theta^2}{\Phi^2}\big),$$

we obtain

$$\ddot{\theta} + \kappa^2 \sin\theta = f_{ml}^{esc}(\theta)\dot{\theta} + \delta_M(\theta) \tag{9}$$

In the approximation when $\sin\theta \approx \theta$, $0 \leq \theta < \pi/3$ equation (9) takes the form

$$\ddot{\widetilde{\theta}} + \kappa^2\widetilde{\theta} = f_{ml}^{esc}(\widetilde{\theta})\dot{\widetilde{\theta}} + \delta_M(\widetilde{\theta}). \tag{10}$$

The solution of this equation with the initial states $\widetilde{\theta}(0) = \theta(0) = \theta_0$, $\dot{\widetilde{\theta}}(0) = \dot{\theta}(0) = 0$ can be represented as

$$\widetilde{\theta}(t) = \theta_0\cos(\kappa t) + g_1(t, \kappa^{esc}, M),$$

$$\dot{\widetilde{\theta}}(t) = -\theta_0\kappa\sin(\kappa t) + g_2(t, \kappa^{esc}, M).$$

For small κ^{esc} and $1/M$ for the time T of second crossing of the trajectory $(\widetilde{\theta}, \dot{\widetilde{\theta}})$ of the straight line $\dot{\theta} = 0$, we approximately obtain

$$T \approx \widetilde{T} = \frac{2\pi}{\kappa} : \; \dot{\widetilde{\theta}}(T) = \dot{\widetilde{\theta}}(0) = 0$$

Let us consider now the Lyapunov function $V(\dot{\widetilde{\theta}}, \widetilde{\theta})$ and its derivative in virtue of system (10), i.e.

$$V(\dot{\widetilde{\theta}}, \widetilde{\theta}) = \dot{\widetilde{\theta}}^2/2 + \frac{\kappa}{2}\widetilde{\theta}^2 > 0$$

and

$$\dot{V}(\dot{\widetilde{\theta}}, \widetilde{\theta}) = \dot{\widetilde{\theta}}(\ddot{\widetilde{\theta}} + \kappa\widetilde{\theta}) = \dot{\widetilde{\theta}}^2 f_{ml}^{esc}(\widetilde{\theta}) + \dot{\widetilde{\theta}}\delta_M(\widetilde{\theta}),$$

respectively. Following [Leonov, 2006], we then estimate the following relation $V(\dot{\widetilde{\theta}}(T), \widetilde{\theta}(T))$ -

$$V(\dot{\widetilde{\theta}}(0), \widetilde{\theta}(0)) = \int\limits_0^T \dot{V}(\dot{\widetilde{\theta}}(t), \widetilde{\theta}(t))\mathrm{d}t \approx$$

$$\approx \int\limits_0^{\widetilde{T}} \dot{V}(\dot{\widetilde{\theta}}(t), \widetilde{\theta}(t))\mathrm{d}t.$$

For this purpose we integrate $\dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big)$ from 0 to $\widetilde{T}$:

$$\int\limits_0^{\widetilde{T}} \dot{\widetilde{\theta}}^2 \kappa_{ml}^{esc}\big(1 - \frac{\widetilde{\theta}^2(\tau)}{\Phi^2}\big)\mathrm{d}\tau = \kappa_{ml}^{esc}\theta_0^2\kappa^2 \int\limits_0^{\widetilde{T}} \sin^2(\kappa\tau)\big(1-$$
$$-\frac{\theta_0^2\cos^2(\kappa\tau)}{\Phi^2}\big)\mathrm{d}\tau + \widetilde{g}_1(\kappa^{esc}, M) =$$
$$= \kappa_{ml}^{esc}\theta_0^2\kappa^2\frac{\pi(4\Phi^2 - \theta_0^2)}{4\kappa\Phi^2} + \widetilde{g}_1(\kappa^{esc}, M)$$

where

$$\lim_{\kappa^{esc},1/M\to 0}\widetilde{g}_1(\kappa^{esc},M)=0.$$

For the approximate values $\cos\widetilde{\theta}(t)\approx\cos\big(\theta_0\cos(\kappa t)\big)$, $\sin\big(2\widetilde{\theta}(t)\big)\approx\sin\big(\theta_0 2\cos(\kappa t)\big)$, the following relations hold

$$\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}(t)\ddot{\widetilde{\theta}}(t)\cos^2(\widetilde{\theta}(t))\mathrm{d}t=\widetilde{g}_2(\kappa^{esc},M)$$

$$\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}^3(t)\sin(2\widetilde{\theta}(t))\mathrm{d}t=\widetilde{g}_3(\kappa^{esc},M),$$

$$\lim_{\kappa^{esc},1/M\to 0}\widetilde{g}_i(\kappa^{esc},M)=0.$$

Taking into account these relations, we integrate $\dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)$ from 0 to $\widetilde{T}$

$$\begin{aligned}
&\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}(t)\delta_M\big(\widetilde{\theta}(t)\big)\mathrm{d}t=\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}(t)\varepsilon_M\big[\ddot{\widetilde{\theta}}(t)2\cos^2\widetilde{\theta}(t)-\\
&-\dot{\widetilde{\theta}}^2(t)\sin 2\widetilde{\theta}(t)\big]\mathrm{d}t=\int_0^{\widetilde{T}}-\theta_0\kappa\sin(\kappa t)\varepsilon_M\\
&\big[-\theta_0\kappa^2\cos(\kappa t)2\cos\big(\theta_0\cos(\kappa t)\big)-\\
&-\theta_0^2\kappa^2\sin^2(\kappa t)\sin\big(\theta_0 2\cos(\kappa t)\big)\big]\mathrm{d}t+\widetilde{g}_4(\kappa^{esc},M)=\\
&=0+g_4(\kappa^{esc},M),
\end{aligned}$$

where

$$\lim_{\kappa^{esc},1/M\to 0}g(\kappa^{esc},M)=0.$$

Then we receive

$$\begin{aligned}
&V(\dot{\widetilde{\theta}}(T),\widetilde{\theta}(T))-V(\dot{\widetilde{\theta}}(0),\widetilde{\theta}(0))\approx\\
&\approx\int_0^{\widetilde{T}}\dot{V}(\dot{\theta}(\tau),\theta(\tau))\mathrm{d}\tau=\\
&=\int_0^{\widetilde{T}}\big[\dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big)+\dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)\big]\mathrm{d}\tau\\
&=\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}^2 f_{ml}^{esc}\big(\widetilde{\theta}(\tau)\big)\mathrm{d}\tau+\int_0^{\widetilde{T}}\dot{\widetilde{\theta}}\delta_M\big(\widetilde{\theta}(\tau)\big)\mathrm{d}\tau=\\
&=\kappa_{ml}^{esc}\theta_0^2\kappa^2\frac{\pi(4\Phi^2-\theta_0^2)}{4\kappa\Phi^2}+g(\kappa^{esc},M),
\end{aligned}$$

where

$$\lim_{\kappa^{esc},1/M\to 0}g(\kappa^{esc},M)=0.$$

In this case for $-2\Phi<\theta_0<2\Phi$ we obtain an unwinding and for $2\Phi<|\theta_0|$ twisting of the phase trajectory for sufficiently small $\kappa^{esc},1/M$. Thus proved the existence of an inphase regime, i.e., we proved that for $\phi_1-\phi_2=0$, for the sum of the angles $\phi_1+\phi_2$ there occurs a periodic regime.

References

[Bennet *et. al.,* 2005] Bennet, M., Schatz, M.F., Rockwood, H.& Wiesenfeld, K. [2005] "Huygens' clock", *Royal society* **1**, 1, pp. 1–18.

[Blekhman 1988] I. I. Blekhman, Synchronization in Science and Technology ASME, New York, 1988.

[Huygens 1669] Huygens, C. 1669 Instructions concerning the use of pendulum-watches, for finding the Longitude at Sea. Philo.T rans. Royal Soc. 4, No 47, 937.

[Huygens 1986] Huygens, C. (translated by R. Blackwell) Christiaan Huygens' The pendulum clock or geometrical demonstrations concerning the motion of pendula as applied to clocks. Ames, IA: Iowa State University Press.

[Leonov, 2006] Leonov, G.A. [2006] Control Theory. Saint-Petersburg: Izd. SPbGU. [in Russian]

[Oud *et. al.,* 2006] W. Oud, H. Nijmeijer, A. Pogromsky [2006] "Experimental results on Huygens synchronization", Proceedings of 1st IFAC Conference on Analysis and Control of Chaotic Systems, 2006, Reims - France

[Pantaleone, 2002] Pantaleone, J. [2002] "Synchronization of metronomes", *Am. J. Phys.* **70**, 10, pp. 992–1000.

[Pogromsky *et. al.,* 2005] A. Y. Pogromsky, V.N. Belykh, H. Nijmeijer [2006] "A study of controlled synchronization of Huijgens pendula", in Group Coordination and Cooperative Control, Editors: K.Y.Pettersen, J.T.Gravdahl, and H.Nijmeijer, 205-216, Springer Verlag, Book Chapter ISBN 3-540-33468-8