2005 International Conference on Physics and Control, PhysCon 2005, Proceedings Volume 2005, 2005, art. num. 1514053, pp. 596-599
(doi:10.1109/PHYCON.2005.1514053)

On stability by the first approximation for discrete systems

Kuznetsov N.V., Leonov G.A.

Draft [1] [2] [3]

Abstract: The problem of stability by the first approximation for discrete systems is considered. The Malkin-Massera-Chetaev theorem on stability by the first approximation is extended to discrete systems. The stability conditions by the first approximation for cascade are obtained.

Keywords: Lyapunov exponent; Lyapunov exponents sign inversion; Characteristic exponent; Lyapunov characteristic exponent; chaos; time-varying linearization; nonstationary; stability by the first approximation; instability; Perron effects; counterexample; strange attractor

1 Introduction

Discrete control systems are widely applied in radiotechnology and communication [1–11]. In the past decades the computer control systems are widely used. The systems are discrete [12–19].

For just this reason the development of methods of stability for discrete systems by the first approximation is the actual problem in control systems.

In the noncritical cases the problem of justification of stability by the first approximation is completely solved for stationary and periodic linearizations, where the stability or instability of solution of nonlinear system is completely defined by the stability or instability of the first approximation system [20–24].

In 1930 year for nonperiodic linearizations O. Perron has found the effects of Lyapunov exponent sign inversion (called later *Perron effect* by the authors) when carried out the linearization procedure. He showed that the negativeness of the largest Lyapunov exponent (Lyapunov characteristic exponent) of the first approximation system does not always result in the stability of zero solution of the original system. In addition, in arbitrary small neighborhood of zero solution the solutions of the original system with positive Lyapunov exponent (Lyapunov characteristic exponent) can exist. For discrete systems the similar effects are described in [26,27,28].

Thus it turns out that the negativeness of Lyapunov exponent (characteristic exponents) of linearized system does not assure the asymptotic stability of the original system.

The first sufficient conditions of asymptotic stability by the first approximation for nonstationary linearizations were obtained by A.M. Lyapunov [20]. He introduced the notion of regular linear system and showed that for regular linearizations the stability is defined by the negativeness of Lyapunov exponents (Lyapunov characteristic exponents) of linearized system. This theorem of Lyapunov was generalized further by Persidskii,Malkin, Massera, Chetaev [31, 32,33, 34].

---

[1] Nikolay V. Kuznetsov, nkuznetsov239 at gmail.com (correspondence author)

[2] PDF slides http://www.math.spbu.ru/user/nk/PDF/Lyapunov-exponent-Sign-inversion-Perron-effects-Chaos.pdf

[3] Extended material: G.A. Leonov, N.V. Kuznetsov, Time-Varying Linearization and the Perron effects, International Journal of Bifurcation and Chaos, Vol. 17, No. 4, 2007, pp. 1079-1107 (doi:10.1142/S0218127407017732)

Kuznetsov N.V., Leonov G.A., Criteria of stability by the first approximation for discrete nonlinear systems, Vestnik St.Petersburg University. Mathematics. Vol. 38, No. 3, 2005, pp. 21-30

Kuznetsov N.V., Leonov G.A., Criterion of stability to first approximation of nonlinear discrete systems, Vestnik St.Petersburg University. Mathematics. Vol. 38, No. 2, 2005, pp. 52-60

Kuznetsov N.V., Leonov G.A., Stability by the first approximation for discrete systems, Vestnik St.Petersburg University. Mathematics. Vol. 36, No. 1, 2003, pp. 21-27

In [29] is shown that the uniform with respect to initial data negativeness of the Lyapunov exponents of the first approximation systems is a sufficient condition of asymptotic stability. Thus it was shown that the Perron effects of the largest Lyapunov exponent sign inversions are only possible on the boundary of flow stable by the first approximation.

In the present paper the theorems of Malkin-Massera-Chetaev and the results of the work [29] are extended to the discrete systems of the form

$$\begin{gathered} x(t+1) = A(t)x(t) + g(t, x(t)),\ x(t) \in \mathbb{R}^n \\ \det A(t) \neq 0 \\ t = 0, 1, \dots, \end{gathered} \tag{1}$$

where $g(\cdot,\cdot)$ is a vector function, mapping $\mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$.

2 Characteristic exponents, Lyapunov exponents, and regular systems

Consider the vector functions $f(t)$ such that $\overline{\lim\limits_{t\to\infty}}|f(t)| \neq 0$.

Definition 1. *The number (or one of the symbols $-\infty$ and $+\infty$), defined by formula $\mathcal{X}[f(t)] = \overline{\lim\limits_{t\to\infty}} \frac{1}{t} \ln |f(t)|$, is called Lyapunov characteristic exponent of vector function $f(t)$.*

Here $|\cdot|$ is an Euclidean norm.

Consider a linear discrete system

$$\begin{gathered} y(t+1) = A(t)y(t), \quad y(t) \in \mathbb{R}^n \\ \det A(t) \neq 0 \\ t = 0, 1, \dots, \end{gathered} \tag{2}$$

where A(t) is a matrix of dimension $n \times n$. Suppose, $Y(t) = (y^1(t), ..., y^n(t))$ is a fundamental matrix of system (2), and σ_Y is a sum of Lyapunov characteristic exponents of its columns $y^i(t)$.

For discrete systems we have [30, 27] the following

Lemma 1 *(the Lyapunov inequality)* **.**

Suppose, all the solutions of system (2) have the characteristic exponents $< +\infty$ (or all characteristic exponents $> -\infty$). Then for any fundamental system of solutions $Y(t)$ the following inequality holds

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)| \leq \sigma_Y. \tag{3}$$

Lemma 2. *If for the matrix of linear system (2) restrictions (a) and (b):*

$$\begin{aligned} &(a) \quad \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |\prod_{j=1}^{t} A(t-j)| \ < +\infty, \\ &(b) \quad \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |\prod_{j=1}^{t} A(t-j)^{-1}| \ < +\infty, \end{aligned}$$

are valid, then each nontrivial solution of system (2) has a finite Lyapunov characteristic exponent.

The result, stated in [30], is a corollary of this lemma.

Corollary. *If there exists the number C such that the following conditions*

$$\sup_t |A(t)| \leq C < +\infty, \quad \sup_t |A(t)^{-1}| \leq C < +\infty$$

are satisfied, then each nontrivial solution of system (2) has a finite Lyapunov characteristic exponent.

Proof of lemma. For any solution $y(t)$ of system (2) the following estimate

$$|y(t)| \leq |y(t_0)| \, |\prod_{j=1}^{t} A(t-j)|$$

is valid. From this and condition (a) of lemma, for all nonzero solutions of system (2) we have $\mathcal{X}[y(t)] < +\infty$ and, therefore, the condition of lemma (1) is satisfied. Then for system (3) by the Lyapunov inequality we obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det Y(j)| < +\infty.$$

Carrying out the similar estimates for the conjugate system

$$z(t+1) = A^*(t)^{-1} z(t), \tag{4}$$

we obtain the assertion of lemma. ■

Definition 2. *A set of distinctive Lyapunov characteristic exponents, of all solutions of discrete system, different from $\pm\infty$ is called a spectrum of it.*

Note, that the number of distinctive Lyapunov characteristic exponents is bounded by a dimension of space.

Definition 3. *A fundamental matrix is said to be normal if the sum of Lyapunov characteristic exponents of its columns is minimal in comparison with other fundamental matrices.*

It is well known [30; 28] the following

Lemma 3 *(on Lyapunov characteristic exponents of normal fundamental system)***.**

1) In all normal fundamental systems of solutions the number of solutions with the same Lyapunov characteristic exponent is equal.

2) Each normal fundamental system realizes the entire spectrum of linear system.

Definition 4. *A set of Lyapunov characteristic exponents $\alpha_1, ...\alpha_n$ of a certain normal fundamental system of solutions $y^1(t), ..., y^n(t)$ is called an entire spectrum of linear system (2) and the number $\sigma = \sum_1^n \alpha_i$ is a sum of Lyapunov characteristic exponents of linear system.*

Definition 5. *A discrete linear system is said to be regular if for the sum of its Lyapunov characteristic exponents σ the relation holds*

$$\sigma = \underline{\lim_{t\to\infty}} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)|.$$

Definition 6. *The number*

$$\Gamma = \sigma - \underline{\lim_{t\to\infty}} \frac{1}{t} \ln \prod_{j=0}^{t-1} |\det A(j)|$$

is called an irregularity coefficient of linear system (2).

Definition 7. *The singular numbers $\{\alpha^j(Y(t))\}_1^n$ of the matrix $Y(t)$ are the square roots of eigenvalues of the matrix*

$$Y(t)^* Y(t).$$

Definition 8. *The Lyapunov exponent μ^j is the number*

$$\mu^j = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \alpha^j(Y(t)).$$

Lemma 4. *The largest Lyapunov characteristic exponent of linear system is equal to the largest Lyapunov exponent.*

The proof of this assertion in detail can be found in [29].

3. Stability criteria by the first approximation

Let $Y(t)$ be a fundamental matrix of linear system

$$y(t+1) = A(t)y(t) \quad t = 0, 1, \dots. \tag{5}$$

Then the solution of system (1) can be represented as

$$x(t) = Y(t)(x(0) + \sum_{\tau=0}^{t-1} Y(\tau+1)^{-1} g(\tau, x(\tau)),\ t = 1, 2\dots. \tag{6}$$

Theorem 1. *Suppose, for the matrix $A(t)$ conditions (a),(b),(c) are valid:*

$$\begin{aligned} &(a) \quad \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |\prod_{j=1}^{t} A(t-j)| \qquad < +\infty, \\ &(b) \quad \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |\prod_{j=1}^{t} A(t-j)^{-1}| \quad < +\infty, \end{aligned} \tag{7}$$

(c) *the linear system 5 is regular and its characteristic exponents are negative.*

If there exists $\kappa > 0$ such that if

$$|g(t, x)| \le \kappa |x|^m \quad (m > 1) \tag{8}$$

then the solution $x(t) \equiv 0$ of system (1) is asymptotically stable.

Theorem 2. *Let there exist the positive numbers C and κ, the neighborhood of zero $\Omega(0)$, and the bounded sequence $p(s)$ such that the conditions*

$$|g(t, x)| \le \kappa |x|, \quad \forall t \ge 0,\ \forall x \in \Omega(0), \tag{9}$$

$$|Y(t)Y(\tau)^{-1}| \le C \prod_{\tau}^{t-1} p(s), \quad \forall t > \tau > 0, \tag{10}$$

and the inequality

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \prod_{0}^{t-1} (p(s) + C\kappa) < 0 \tag{11}$$

are satisfied. Then the solutions $x(t) \equiv 0$ of system (1) is asymptotically stable by Lyapunov.

Corollary. *For the systems of the first order the negativeness of Lyapunov exponent (Lyapunov characteristic exponent) of the system of the first approximation results in the asymptotic stability of zero solution, i.e. for one-dimensional systems there is no Perron effects of Lyapunov exponent sign inversions.*

Theorem 3. *If the conditions*

$$\begin{gathered} |Y(t)Y(\tau)^{-1}| \le C \exp[-\alpha(t-\tau) + \gamma\tau], \\ \forall t \ge \tau \ge 0, \\ \alpha > 0,\ \gamma \ge 0, \end{gathered} \tag{12}$$

$$\begin{gathered} |g(t, x)| < \kappa |x|^\nu, \quad \forall x \in \Omega_0,\ t = 0, 1, \dots, \\ \kappa > 0, \nu > 1 \end{gathered} \tag{13}$$

where κ is sufficiently small,

and the inequality

$$(\nu - 1)\alpha - \gamma > 0 \tag{14}$$

are satisfied, then the solution of systems (1) is asymptotically stable by Lyapunov.

Remark. The estimate for the norm of the Cauchy matrix can be obtained by means of the Lyapunov characteristic exponents and irregularity coefficient.

Proof of Theorem 1.

By condition (c) of theorem 1 there exists the positive number $a > 0$ such that for entire spectrum of system (5) the following estimate

$$a_j < -a < 0, \quad j = 1..n$$

is valid. The fact that by condition (c) system (5) is regular implies that in the case $\tau < t$ for its Cauchy matrix it is well known the following estimate [30; 28]

$$|Y(t)Y^{-1}(\tau)| \leq ce^{-a(t-\tau)}e^{\varepsilon\tau}.$$

Introducing the denotation $|x(t)|e^{at} = y(t)$, we obtain

$$|y(t)| \leq c_Y|y(0)| + \sum_{\tau=0}^{t-1} c_0\kappa e^{(\varepsilon+a)}e^{\tau(\varepsilon-a(m-1))}|y(\tau)|^m. \tag{15}$$

Since (15) is valid for any $\varepsilon > 0$, we may choose ε such that the following inequality $(\varepsilon - a(m-1)) < 0$. is satisfied. Then, taking into account the lemma [27], resulted from a discrete analog of Gronwall's lemma, we obtain

$$|y(t)| < 1 \quad \forall t = 0, 1...$$

This implies that $|x(t)|$ is bounded and $|x(t)| \to 0$ as $t \to +\infty$. ■

Remark. *If in place (7) we make use of more restrict limitative conditions, namely*

$$\sup_t |A(t)| \leq C < +\infty, \quad \sup_t |A(t)^{-1}| \leq C < +\infty,$$

then we obtain the result, stated in [30].

Proof of Theorem 2.

The condition (10) of theorem yields the estimate

$$|Y(t)| \leq C\prod_0^{t-1} p(s), \quad \forall t \geq \tau \geq 0.$$

The condition (11) of theorem implies that there exists the number N such that

$$\frac{1}{t}\ln\prod_0^{t-1}(p(s) + C\kappa) < 0 \quad \forall t > N.$$

Then by the boundedness of $p(s)$ there exists a set of initial data such that

$$x(t, x_0) \in \Omega(0) \quad \forall t < N.$$

By (6) and condition (10) of theorem the estimate holds

$$|x(t)| \leq C\prod_0^{t-1} p(s)|x(0)| + C\sum_{\tau=0}^{t-1}(\prod_{\tau+1}^{t-1} p(s))\kappa|x(\tau)|.$$

Applying a discrete analog of Gronwall's lemma, we obtain the following estimate

$$|x(t)|\prod_{s=0}^{t-1} p(s)^{-1} < C|x(0)|\prod_{s=0}^{t-1}(C\kappa p(s)^{-1}+1).$$

From conditions (11) of theorem it follows that $|x(t)| \to 0$, as $t \to +\infty$. ■

Proof of Theorem 3.

By (12) we obtain

$$|Y(t)Y(\tau+1)^{-1}| \le C\exp[-\alpha(t-\tau)+\gamma\tau]e^{\alpha+\gamma}$$

Then estimate (13) and (6) imply

$$\begin{aligned}&(e^{\alpha t}|x(t)|) \le \\ &\le C|x(0)| + Ce^{\alpha+\gamma}\kappa\sum_{\tau=0}^{t-1} e^{(\alpha(1-\nu)+\gamma)\tau}(e^{\alpha\tau}|x(\tau)|)^{\nu}\end{aligned}$$

Since by condition (14) of theorem we have $(\alpha(1-\nu)+\gamma) < 0$, for sufficiently small $|x(0)|$ the following relation

$$(e^{\alpha t}|x(t)|) \le 1$$

is valid. ■

3 The stability of solutions with respect to a set of initial dates

Consider a discrete system

$$\begin{gathered}x(t+1) = F(t, x(t)),\ x(t) \in \mathbb{R}^n, \\ t = 0, 1, 2, ...,\end{gathered} \tag{16}$$

where $F(\cdot,\cdot)$ is a twice continuously differentiable in the second argument vector-function.

Consider a linearization of system (16) along the solutions with the initial data x_0 from the open and bounded in $\mathbb{R}^n$ set Ω, namely

$$y(t+1) = \left.\frac{\partial F(t,x)}{\partial x}\right|_{x=x(t,x_0)} y(t), \tag{17}$$

where $x(0,x_0) = x_0$. Let for the largest singular number of system (17) the following estimate

$$\alpha^1(t,x_0) < \alpha(t), \quad \forall x_0 \in \Omega,\ t = 0, 1, 2, ... \tag{18}$$

where $\alpha(t)$ is a scalar function, be valid. Consider also the fundamental matrix of linear system (17), namely $Y(t,x_0), \quad Y(0,x_0) = I$.

Theorem 4. *If there exists the bounded $\alpha(t)$ such that estimate (18) is satisfied, then the solution $x(t,x_0)$, where $x_0 \in \Omega$, is stable by Lyapunov.*

If, in addition, $\lim\limits_{t\to\infty} \alpha(t) = 0$, then the solution is asymptotically stable by Lyapunov.

Corollary. *The effects of Perron are only possible on the boundary of cascade stable by the first approximation.*

Proof of Theorem 4.

Under the assumptions, imposed on the functions $F(\cdot,\cdot)$ [35], for the vectors y, z and the numbers $t \ge 0$ there exists the vector w such that the following relations

$$|w-y| \le |y-z|, \tag{19}$$

$$|x(t,y)-x(t,z)| \le \left|(x(t,w))'_w\right| |y-z| \tag{20}$$

are satisfied.

Since $\frac{\partial x(t,h)}{\partial h} = Y(t,h)$, then from (18) by (19), (20) we obtain that for any y_0 such that

$$\{w|\ |w - x_0| \le |x_0 - y_0|\} \subset \Omega$$

and for all $t \ge 0$ the estimate holds

$$\begin{gathered} |x(t,x_0) - x(t,y_0)| \le \\ \le |x_0 - y_0| \sup \alpha^1(t,w) \le \alpha(t)|x_0 - y_0|, \end{gathered} \tag{21}$$

where the supremum is taken over all w from the ball

$$\{w|\ |w - x_0| \le |x_0 - y_0|\}.$$

From estimate (21) follows the assertion of theorem. ■

References

[1] V.A. Besekerskii, *Digital automatic systems.* M.: Nauka, 1976. [In Russian]

[2] M.P. Bobnev, B.Kh. Krivitskii, M.S. Yarlukov, *Complex systems of radioautomatics (in Russian),* M.: Sov. Radio, 1968. [In Russian]

[3] P.N. Vidal', *Nonlinear pulse systems.* M.: Energia, 1974. [In Russian]

[4] E. Djuri, *Pulse systems of automatic control [Russian translation]*, M.: Fizmatgis, 1963.

[5] R. Izerman, *Digital control systems.* M.: Mir, 1984. [In Russian]

[6] G.A. Leonov, V.B. Smirnova, *Mathematical problems of the theory of phase lock,* St. Petersburg: Nauka, 2000. (in Russian)

[7] Ya.Z. Tsypkin, *Theory of pulse system,* M.: Fizmatgis, 1958. (in Russian)

[8] Ya.Z. Tsypkin, *The foundation of the theory of automatic systems* , M.: Nauka, 1977. (in Russian)

[9] Ya.Z. Tsypkin, Yu.S. Popkov, *Theory of nonlinear pulse systems*, M.: Nauka, 1973. (in Russian)

[10] V.V. Shakhtildyan, A.A. Lyakhovkin, *Phase-locked loops,* M.: Svyaz', 1972. (in Russian)

[11] *Phase-locked loops with discretization elements/ Editor V. V. Shakhtildyan* , M.: Svyaz', 1979. (in Russian)

[12] V.A. Besekerskii, V.V. Izrantsev, *Automatic control systems with microcomputers.* M.: Nauka, 1987,

[13] B.P. Kogan, V.V. Stashin, *Microprocessors in digital systems.* M.: Energia, 1979. (in Russian)

[14] B. Kuo, *Theory and design of digital control systems,* M.: Mashinostroenie, 1986. (Russian translation)

[15] G.A. Leonov, S.M. Seledzhi, *Phase-locked loops in analog and digital circuitry*, St. Petersburg: "Nevskii Dialect", 2002. (in Russian)

[16] T.G. Li, G.A. Adams, W.M. Geinz, *Computer process control. Modelling and optimization [Russian translation],* M.: Sov. Radio, 1972.

[17] *Microcomputers,* / Editor A.Dircsen [Russian translation], M.: Energoizdat, 1982.

[18] V.M. Mutter, *Analog-digital automatic systems: Design and computing.* L.: Mashinostroenie, 1981.

[19] K. Dstrem, B. Vittermark, *Computer control system [Russian translation]*, M.: Mir, 1987.

[20] A.M. Lyapunov, *General problem of motion stability*, Khar'kov, 1892, 250 p. (in Russian)

[21] B.P. Demidovich, *The lectures on the mathematical theory of stability*, Moscow, 1967. (in Russian)

[22] J.P. LaSalle, *The Stability and Control of Discrete Processes*, Springer-Verlag, 1980, 150 p.

[23] S. Elyadi, *An introduction to difference equations*, Berlin: Springer, 1996, 389 p.

[24] V.P. Bromberg, *Matrix methods in the theory of relay and impulse control,* 1986. M., Nauka, 321 p.

[25] O. Perron, Die Stabilitatsfrage bei Differentialgleichungen, *Mathematische Zeitschrift*, 1930, bd.32, N 5, p. 702–728

[26] N.V. Kuznetsov, G.A. Leonov, *Counterexample of Perron in the Discrete Case*, *Izv. RAEN, Diff. Uravn.*, N5, 2001, pp. 71 (in Russian)

[27] N.V. Kuznetsov, G.A. Leonov, Stability by the first approximation of discrete systems, *Vestnik Sankt-Peterburgskogo Universiteta. Ser 1. Matematika Mekhanika Astronomiya*, No. 1, pp. 28-35, 2003.

[28] I.V. Gaishun, *Discrete-time systems*, Minsk, Inst. matemat. AN Belarus', 2001, 400 p. (in Russian)

[29] G.A. Leonov, *Strange attractors and the classical theory of motion stability*, Izd. SPbGU, 2004, 142 p. (in Russian)

[30] V.B. Demidovich, A stability criterion for difference equations, *Diff. Uravneniya*, Vol. 5, No 7, 1969, pp. 1247–1255.

[31] K.P. Persidskii, Characteristic numbers of differential equations, *Izv. AN Kazach. SSR*, 1947, Math. and Mech., issue 1.

[32] I.G. Malkin, *Theory of motion stability*, M., Nauka, 1966, 525P. (in Russian)

[33] Kh. L. Massera, Theory of stability [Russian translation], *Collection of translations "Matematica", IL*, (1957), pp. 81–101 .

[34] N.G. Chetaev, *Motion stability*, M., Gostekhizdat, 1955, 207 p. (in Russian)

[35] V.A. Zorich, *Mathematical analysis. P. I, II.*, M.: Nauka, 543 p.; 1984, 640 p. (in Russian)