HAAR SPECTRA OF DISCRETE SIGNALS[1]

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This paper is a development of works [1, 2].

We will refer to a vector space $\mathbb{R}_N$ of real N-periodic sequences $x = \{x(j)\},\ j \in \mathbb{Z}$, as *a space of signals.* Denote by $\delta_N(j)$ an N-periodic unit impulse, which is equal to 1 when $j = 0, \pm N, \pm 2N, \ldots$ and is equal to 0 on other indices $j \in \mathbb{Z}$. In a space $\mathbb{R}_N$ with $N = 2^s$ consider two wavelet bases:

$$c(j); \quad f_\nu(j - p\Delta_{\nu+1}), \quad p = 0, 1, \ldots, N_\nu - 1, \quad \nu = 1, \ldots, s; \tag{1}$$

$$c(j); \quad g_\nu(j - p), \quad p = 0, 1, \ldots, N_\nu - 1, \quad \nu = 1, \ldots, s. \tag{2}$$

Here $\Delta_\nu = 2^{\nu-1},\ N_\nu = N/2^\nu,\ c(j) \equiv N^{-1}$,

$$f_\nu(j) = \frac{1}{2^\nu}\Big\{\sum_{q=0}^{\Delta_\nu - 1} \delta_N(j-q) - \sum_{q=\Delta_\nu}^{\Delta_{\nu+1}-1} \delta_N(j-q)\Big\},$$

$$g_\nu(j) = 2^{-\nu}\left\{\delta_{N_{\nu-1}}(j) - \delta_{N_{\nu-1}}(j - N_\nu)\right\}.$$

Basis (1) is referred to as *Haar basis related with subsampling in frequency.* It is well-known (extensive bibliography may be found in [3]). Basis (2) is referred to as *Haar basis related with subsampling in time.* It was studied in [2].

Any signal $x \in \mathbb{R}_N$ can be expanded over orthogonal basis (1):

$$x(j) = \alpha N^{-1} + \sum_{\nu=1}^{s} \sum_{p=0}^{N_\nu - 1} x_\nu(p)\, f_\nu(j - p\Delta_{\nu+1}), \quad j \in \mathbb{Z}.$$

The set of coefficients of this expansion $H_f(x) = \{\alpha;\ x_\nu(p),\ p \in 0 : N_\nu - 1,\ \nu \in 1 : s\}$ forms the *Haar spectrum of signal x related with subsampling in frequency.* Similarly, the signal can be expanded over orthogonal basis (2):

$$x(j) = \alpha' N^{-1} + \sum_{\nu=1}^{s} \sum_{p=0}^{N_\nu - 1} x'_\nu(p)\, g_\nu(j - p), \quad j \in \mathbb{Z}.$$

The set of coefficients of this expansion $H_g(x) = \{\alpha';\ x'_\nu(p),\ p \in 0 : N_\nu - 1,\ \nu \in 1 : s\}$ forms the *Haar spectrum of signal x related with subsampling in time.*

We start with basis (2). The following notations will be permanently used: $\lfloor \xi \rfloor$ means the whole part of a real number ξ; $\langle k \rangle_n = k - \lfloor k/n \rfloor n$ means the reminder of an integer k after division by n.

Theorem 1. *Fix $q \in 0 : N-1$ and $x \in \mathbb{R}_N$. Suppose that $H_g(x) = \{\alpha;\ x_\nu(p)\}$ and $H_g\big(x(\cdot - q)\big) = \{\widetilde{\alpha};\ \widetilde{x}_\nu(p)\}$. Then $\widetilde{\alpha} = \alpha$ and*

$$\widetilde{x}_\nu(p) = (-1)^{\lfloor (q-p-1)/N_\nu \rfloor + 1}\, x_\nu(\langle p - q \rangle_{N_\nu}), \qquad p \in 0 : N_\nu - 1, \quad \nu \in 1 : s.$$

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Theorem 2. *Fix $x, y \in \mathbb{R}_N$. Let $z \in \mathbb{R}_N$ be a circular convolution of signals x and y (a signal with components $z(j) = \sum_{k=0}^{N-1} x(k)\, y(j-k)$, $j \in \mathbb{Z}$). Suppose that $H_g(x) = \{\alpha;\ x_\nu(p)\}$, $H_g(y) = \{\beta;\ y_\nu(p)\}$ and $H_g(z) = \{\gamma;\ z_\nu(p)\}$. Then $\gamma = \alpha\beta$ and*

$$z_\nu(p) = \sum_{q=0}^{p} x_\nu(q)\, y_\nu(p-q) - \sum_{q=p+1}^{N_\nu-1} x_\nu(q)\, y_\nu(N_\nu + p - q), \qquad p \in 0 : N_\nu - 1, \quad \nu \in 1 : s.$$

Analogous results hold for basis (1) if we change the notion of a shift. Let $p, q \in 0 : N-1$, $p = p_{s-1}2^{s-1} + \cdots + p_0$, $q = q_{s-1}2^{s-1} + \cdots + q_0$. A value $p \oplus q = \sum_{k=0}^{s-1} \langle p_k + q_k \rangle_2\, 2^k$ is called *a dyadic shift* of number p by value q [4].

Theorem 3. *Fix $q \in 0 : N-1$ and $x \in \mathbb{R}_N$. Suppose that $H_f(x) = \{\alpha;\ x_\nu(p)\}$ and $H_f\big(x(\cdot \oplus q)\big) = \{\widetilde{\alpha};\ \widetilde{x}_\nu(p)\}$. Then $\widetilde{\alpha} = \alpha$ and*

$$\widetilde{x}_\nu(p) = (-1)^{q_{\nu-1}} x_\nu(p \oplus \lfloor q/\Delta_{\nu+1} \rfloor), \qquad p \in 0 : N_\nu - 1, \quad \nu \in 1 : s.$$

Theorem 4. *Fix $x, y \in \mathbb{R}_N$. Let $z \in \mathbb{R}_N$ be a dyadic convolution of signals x and y (a signal with components $z(j) = \sum_{k=0}^{N-1} x(k)\, y(j \oplus k)$, $j \in 0 : N-1$). Suppose that $H_f(x) = \{\alpha;\ x_\nu(p)\}$, $H_f(y) = \{\beta;\ y_\nu(p)\}$ and $H_f(z) = \{\gamma;\ z_\nu(p)\}$. Then $\gamma = \alpha\beta$ and*

$$z_\nu(p) = \sum_{q=0}^{N_\nu-1} x_\nu(q)\, y_\nu(p \oplus q), \qquad p \in 0 : N_\nu - 1, \quad \nu \in 1 : s.$$

Define a permutation named *reverse*. Let $j \in 0 : 2^\nu - 1$, $j = j_{\nu-1}2^{\nu-1} + j_{\nu-2}2^{\nu-2} + \ldots + j_0$. Then, by a definition, $\mathrm{rev}_\nu(j) = j_0 2^{\nu-1} + j_1 2^{\nu-2} + \ldots + j_{\nu-1}$.

Haar spectrum $X = \{\alpha;\ x_\nu(p)\}$ is called *logarithmically autoreversal* if it satisfies to the relation

$$x_\nu(p) = x_\nu(\mathrm{rev}_{s-\nu}(p)), \qquad p \in 0 : N_\nu - 1, \quad \nu \in 1 : s-1.$$

The following theorem establishes a link between two types of Haar spectra.

Theorem 5. *Let signals $x, y \in \mathbb{R}_N$ be related by a condition $H_f(y) = H_g(x) =: X$. If spectrum X is logarithmically autoreversal then the signals satisfy to the relation*

$$y(j) = x\big(\mathrm{rev}_s(j)\big), \quad j \in 0 : N-1. \tag{3}$$

This theorem is in a sense invertible.

Theorem 6. *Let signals $x, y \in \mathbb{R}_N$ satisfy to (3) and $H_f(y) = H_g(x) =: X$. Then spectrum X is logarithmically autoreversal.*

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