Adaptive Signal Processing Based on Discrete Vilenkin–Chrestenson Transform

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Abstract: *In a space of discrete N-periodic signals with $N = n^s$, two recurrent sequences of orthogonal bases related with discrete Vilenkin–Chrestenson transform are constructed. These constructions allow to form generalized wavelet packets that can be effectively used in adaptive signal processing.*

Keywords: – Adaptive signal processing, wavelet packets, Vilenkin–Chrestenson transform.

1. DISCRETE VILENKIN–CHRESTENSON FUNCTIONS

We deal with a linear space $\mathbb{C}_N$ of complex-valued discrete N-periodic functions $x = x(j)$, $j \in \mathbb{Z}$. Elements of $\mathbb{C}_N$ are called signals. The notions of scalar product and norm in $\mathbb{C}_N$ are introduced as usual:

$$\langle x, y\rangle = \sum_{j=0}^{N-1} x(j)\,\overline{y(j)}, \quad \|x\| = \langle x, x\rangle^{1/2}.$$

The unit N-periodic impulse $\delta_N(j)$ is a signal which is equal to 1 when j is divisible by N and is equal to 0 on the others $j \in \mathbb{Z}$. The system of shifts $\delta_N(j)$, $\delta_N(j-1), \ldots, \delta_N(j-N+1)$ forms an orthonormal basis in $\mathbb{C}_N$.

We consider the case of $N = n^s$, where $n, s \geq 2$. For any $k \in \{0, 1, \ldots, N-1\}$ we can write its n-adic code: $k = (k_{s-1}, k_{s-2}, \ldots, k_0)_n$. Let as well j be from $\{0, 1, \ldots, N-1\}$, $j = (j_{s-1}, j_{s-2}, \ldots, j_0)_n$. Denote

$$\{k, j\}_s = \sum_{\nu=0}^{s-1} k_\nu j_\nu.$$

The signals

$$v_k(j) = \omega_n^{\{k,j\}_s}, \quad k = 0, 1, \ldots, N-1,$$

where $\omega_n = \exp(2\pi i/n)$, are called *the discrete Vilenkin–Chrestenson functions* [1–3]. The set of signals $v_0, v_1, \ldots, v_{N-1}$ forms an orthogonal basis in $\mathbb{C}_N$.

2. TWO SEQUENCES OF ORTHOGONAL BASES

We construct two sequences of orthogonal bases in $\mathbb{C}_N$ that evolve recurrently from the shifts of a unit N-periodic impulse to discrete Vilenkin–Chrestenson functions [4].

Denote $N_\nu := N/n^\nu = n^{s-\nu}$, $\Delta_\nu = n^{\nu-1}$. The first recurrent sequence $\{f_\nu(k; j)\}_{k=0}^{N-1}$, $\nu = 0, 1, \ldots, s$, is formed as follows:

$$f_0(k) = \delta_N(\cdot - k), \quad k = 0, 1, \ldots, N-1;$$

$$f_\nu(l+\sigma\Delta_\nu+p\Delta_{\nu+1}) = \sum_{\tau=0}^{n-1} \omega_n^{\sigma\tau}\, f_{\nu-1}\big(l+(np+\tau)\Delta_\nu\big), \tag{1}$$

$$l = 0, 1, \ldots, \Delta_\nu - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\sigma = 0, 1, \ldots, n-1, \quad \nu = 1, \ldots, s.$$

The final signals are the discrete Vilenkin–Chrestenson functions: $f_s(k; j) = v_k(j)$, $k = 0, 1, \ldots, N-1$. For every $\nu \in \{0, 1, \ldots, s\}$ the signals $f_\nu(0)$, $f_\nu(1)$, $\ldots$, $f_\nu(N-1)$ form an orthogonal basis in a space $\mathbb{C}_N$, and $\|f_\nu(k)\|^2 = n^\nu$ for all $k = 0, 1, \ldots, N-1$.

Let's clarify the scheme (1). It is assumed that the basis $f_{\nu-1}$ is divided into Δ_ν blocks; these blocks are marked with index l. Every block contains $N_{\nu-1}$ elements with inner indexing $np+\tau$. According to (1) the l-th block of the basis $f_{\nu-1}$ generates n blocks of the basis f_ν with indices $l+\sigma\Delta_\nu$, $\sigma = 0, 1, \ldots, n-1$; every new block contains N_ν elements with inner index p. Fig. 1 illustrates the subdivision scheme (1) when $N = 3^2$. Blocks are shown with squares. A number inside a square indicates the amount of signals contained in the block.

The second recurrent sequence of orthogonal bases $\{g_\nu(k; j)\}_{k=0}^{N-1}$, $\nu = 0, 1, \ldots, s$, is formed as follows:

$$g_0(k) = \delta_N(\cdot - k), \quad k = 0, 1, \ldots, N-1;$$

$$g_\nu\big((nl+\sigma)N_\nu+p\big) = \sum_{\tau=0}^{n-1} \omega_n^{\sigma\tau}\, g_{\nu-1}(lN_{\nu-1}+\tau N_\nu+p), \tag{2}$$

$$l = 0, 1, \ldots, \Delta_\nu - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\sigma = 0, 1, \ldots, n-1, \quad \nu = 1, \ldots, s.$$

As in the previous case, the final signals are the discrete Vilenkin–Chrestenson functions: $g_s(k; j) =$

$= v_k(j)$, $k = 0, 1, \ldots, N-1$. The intermediate signals $g_\nu(k)$, of course, differ from the corresponding signals $f_\nu(k)$. For every $\nu \in \{0, 1, \ldots, s\}$ the signals $g_\nu(0)$, $g_\nu(1)$, $\ldots$, $g_\nu(N-1)$ form an orthogonal basis in a space $\mathbb{C}_N$, and $\|g_\nu(k)\|^2 = n^\nu$ for all $k = 0, 1, \ldots, N-1$.

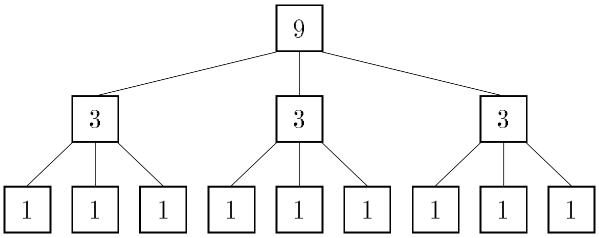

Fig. 1. Subdivision scheme

Let's clarify the scheme (2). It is assumed that the basis $g_{\nu-1}$ is divided into Δ_ν blocks; these blocks are marked with inner index l. Every block contains $N_{\nu-1}$ elements with indexing $\tau N_\nu + p$. According to (2) the l-th block of the basis $g_{\nu-1}$ generates n blocks of the basis g_ν with inner indexing $nl + \sigma$, $\sigma = 0, 1, \ldots, n-1$; every new block contains N_ν elements with index p. Fig. 1 above illustrates the subdivision scheme (2) when $N = 3^2$.

The key property of constructed bases is their block structure. The remarkable fact is that all signals inside a selected block are formed by the shifts of one signal. To be more precise, the following identities are valid:

$$f_\nu(l + p\Delta_{\nu+1}; j) = f_\nu(l; j - p\Delta_{\nu+1}),$$

$$g_\nu(lN_\nu + p; j) = g_\nu(lN_\nu; j - p),$$

$$l = 0, 1, \ldots, \Delta_{\nu+1} - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\nu = 1, \ldots, s.$$

3. FAST VILENKIN–CHRESTENSON TRANSFORMS

Every signal x can be expanded over an orthogonal basis $f_\nu(0)$, $f_\nu(1)$, $\ldots$, $f_\nu(N-1)$ with every $\nu \in \{0, 1, \ldots, s\}$:

$$x = \sum_{k=0}^{N-1} x_\nu(k)\, f_\nu(k).$$

Here $x_\nu(k) = n^{-\nu}\langle x, f_\nu(k)\rangle$. Taking (1) into account we can easily deduce the recurrent relation for the coefficients $x_\nu(k)$:

$$x_0(k) = x(k), \quad k = 0, 1, \ldots, N-1;$$

$$x_\nu(l+\sigma\Delta_\nu+p\Delta_{\nu+1}) = \frac{1}{n}\sum_{\tau=0}^{n-1}\omega_n^{-\sigma\tau}\, x_{\nu-1}\big(l+(np+\tau)\Delta_\nu\big), \tag{3}$$

$$l = 0, 1, \ldots, \Delta_\nu - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\sigma = 0, 1, \ldots, n-1, \quad \nu = 1, \ldots, s.$$

When $\nu = s$, we obtain the expansion of the signal x over the Vilenkin–Chrestenson basis

$$x(j) = \sum_{k=0}^{N-1} x_s(k)\, v_k(j). \tag{4}$$

By the way we compute the expansion coefficients for all intermediate bases f_ν.

Formula (3) can be inverted:

$$x_{\nu-1}\big(l+(np+\tau)\Delta_\nu\big) = \sum_{\sigma=0}^{n-1}\omega_n^{\sigma\tau}\, x_\nu(l+\sigma\Delta_\nu+p\Delta_{\nu+1}), \tag{5}$$

$$l = 0, 1, \ldots, \Delta_\nu - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\tau = 0, 1, \ldots, n-1, \quad \nu = s, s-1, \ldots, 1.$$

The principal role of formula (5) is that it is used to evaluate the samples of the signal x given in the form (4) on the basic period, namely, $x(k) = x_0(k)$, $k = 0, 1, \ldots, N-1$.

Similarly, the signal x can be expanded over an orthogonal basis $g_\nu(0)$, $g_\nu(1)$, $\ldots$, $g_\nu(N-1)$ with every $\nu \in \{0, 1, \ldots, s\}$:

$$x = \sum_{k=0}^{N-1} y_\nu(k)\, g_\nu(k). \tag{6}$$

Here $y_\nu(k) = n^{-\nu}\langle x, g_\nu(k)\rangle$. Taking (2) into account we can easily deduce the recurrent relation for the coefficients $y_\nu(k)$:

$$y_0(k) = x(k), \quad k = 0, 1, \ldots, N-1;$$

$$y_\nu\big((nl+\sigma)N_\nu+p\big) = \frac{1}{n}\sum_{\tau=0}^{n-1}\omega_n^{-\sigma\tau}\, y_{\nu-1}(lN_{\nu-1}+\tau N_\nu+p), \tag{7}$$

$$l = 0, 1, \ldots, \Delta_\nu - 1, \quad p = 0, 1, \ldots, N_\nu - 1,$$

$$\sigma = 0, 1, \ldots, n-1, \quad \nu = 1, \ldots, s.$$

In case $\nu = s$ we obtain

$$x(j) = \sum_{k=0}^{N-1} y_s(k)\, v_k(j),$$

thus $y_s(k) = x_s(k)$ for all $k = 0, 1, \ldots, N-1$. By the way we compute the expansion coefficients for all intermediate bases g_ν which, of course, differ from those for intermediate bases f_ν.

Formula (7) can be inverted:

$$y_{\nu-1}(lN_{\nu-1}+\tau N_\nu+p)=\sum_{\sigma=0}^{n-1}\omega_n^{\sigma\tau}\,y_\nu\big((nl+\sigma)N_\nu+p\big), \tag{8}$$

$$l=0,1,\ldots,\Delta_\nu-1,\quad p=0,1,\ldots,N_\nu-1,$$

$$\tau=0,1,\ldots,n-1,\quad \nu=s,s-1,\ldots,1.$$

Using (8) one can evaluate the samples of the signal x given in the form (6) with some $\nu\in\{1,\ldots,s\}$ on the basic period.

The computational schemes (3) and (7) we call *the fast Vilenkin–Chrestenson transforms related with, respectively, subsampling in frequency and subsampling in time.*

In [3, 4] it is stated that the frequency of a discrete Vilenkin–Chrestenson function v_k is equal to $\mathrm{rev}_s(k)$, where rev_s is a permutation that maps a number $k=(k_{s-1},k_{s-1},\ldots,k_0)_n$ to a number $\mathrm{rev}_s(k)=(k_0,k_1,\ldots,k_{s-1})_n$. Denote $\mathrm{Pal}_k(j)=$ $=v_{\mathrm{rev}_s(k)}(j)$. It is clear that the frequency of function Pal_k is equal to k. Let's rewrite formula (4) in the form

$$x(j)=\sum_{k=0}^{N-1}x_s\big(\mathrm{rev}_s(k)\big)\,\mathrm{Pal}_k(j). \tag{9}$$

It is the expansion (9) that should be used for the digital signal processing, because the basic Vilenkin–Chrestenson functions in this expansion are ordered by frequency.

4. WAVELET PACKETS

Fig. 1 has showed how orthogonal bases f_0, f_1, $\ldots$, f_s and g_0, g_1, $\ldots$, g_s are formed consecutively on horizontal levels $\nu=0,1,\ldots,s$. We can significantly increase the number of orthogonal bases in $\mathbb{C}_N$ if we use the vertical constituent [4]. This is demonstrated in Fig. 2 with $N=3^3$. In all four examples given, the signals in hanging blocks form orthogonal bases. Indeed, the orthogonality of the signals of the ν-th level is known. The signals in every hanging block of the $(\nu+1)$-th level are the linear combinations of signals of some block of the ν-th level, and therefore they are orthogonal to the signals from the other blocks of the ν-th level.

Orthogonal bases in $\mathbb{C}_N$ that are gathered from the blocks of all levels $\nu=1,\ldots,s$ we call *the generalized wavelet bases.* For example, the ramification scheme (a) in Fig. 2 corresponds to Haar–Chrestenson wavelet basis. The set of all wavelet bases is called *a wavelet packet.*

5. ADAPTIVE SIGNAL PROCESSING

Wavelet packets allow to retrieve detailed time-frequency portraits of signals varying resolution in time and frequency. At that sense a wavelet packet analysis can be considered as a powerful extension of the traditional Fourier analysis. On the other hand, using all bases in wavelet packet produces highly redundant representation of a signal. It provides the opportunities of selecting the orthogonal basis adapted to the class of signals or even to a single signal (see [5–7]).

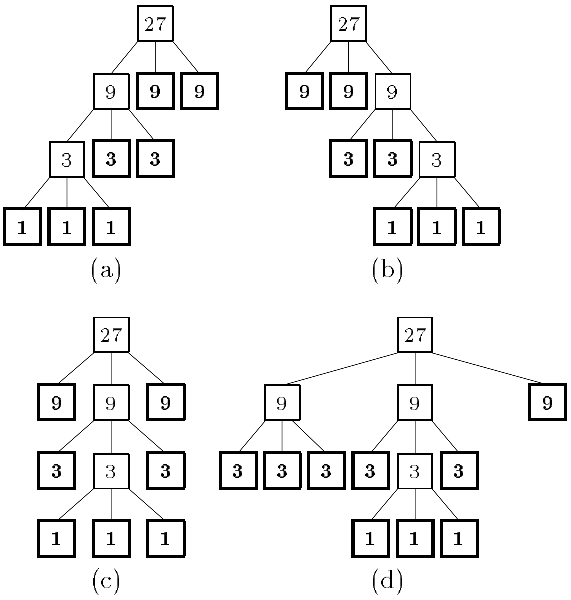

Fig. 2. Ramification patterns

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