On the Least Squares Method over a Discrete Rectangle

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Abstract

The problem of the polynomial LS approximation of two-dimensional discrete data is considered. For solving that problem, the theory of discrete orthogonal polynomials is attracted. Algorithms for constructing the best mean square approximation polinomial and for determining its values are offered. The results of numerical experiments are submitted.

Polynomial approximation of discrete data over a uniform two-dimensional grid is one of the problems appearing in image processing. We shall concern the best mean square approximation.

Denote by Π_{st} the set of algebraic polynomials of two variables of order not higher than (s,t) [1]. Let the real-valued discrete function $f(i,j)$, $i \in 0:M$, $j \in 0:N$, is given. It is required to find a polynomial from Π_{st} $(s \le M,\ t \le N)$ being the solution of the following extremum problem:

$$\sum_{i=0}^{M}\sum_{j=0}^{N}\big[f(i,j)-\pi_{st}(i,j)\big]^2 \longrightarrow \min_{\pi_{st}\in\Pi_{st}}. \tag{1}$$

Introduce the following notations: $\mathcal{P} = \big\{(i,j)\,|\,i \in 0:M,\ j \in 0:N\big\}$ — a discrete rectangle; $\mathcal{F}(\mathcal{P})$ — a set of real-valued functions defined on $\mathcal{P}$.

Introduce the scalar product in $\mathcal{F}(\mathcal{P})$:

$$\langle f,g\rangle = \sum_{i=0}^{M}\sum_{j=0}^{N} f(i,j)\,g(i,j)\,. \tag{2}$$

The problem (1) may be rewritten as

$$\|f-\pi_{st}\|^2 := \langle f-\pi_{st}, f-\pi_{st}\rangle \longrightarrow \min_{\pi_{st}\in\Pi_{st}}. \tag{3}$$

For solving this problem, we shall use the ideas of P. L. Chebyshev [2], generalized for the two-dimensional case. Let $\big\{H_m(x)\big\}_{m=0}^{M}$ and $\big\{V_n(y)\big\}_{n=0}^{N}$ be systems of polynomials that are orthogonal on the sets $0:M$ and $0:N$, respectively, and $\deg H_m = m$, $\deg V_n = n$. Polynomials $\varphi_{mn}(x,y) = H_m(x)\,V_n(y)$ are orthogonal subject to the scalar product (2). We shall look for the solution of the problem (3) in a form

$$\pi_{st}(x,y) = \sum_{m=0}^{s}\sum_{n=0}^{t} a_{mn}\,\varphi_{mn}(x,y)\,. \tag{4}$$

Optimal coefficients a_{mn} are *the Fourier coefficients* of the function f defined by

$$a_{mn} = \frac{\langle f, \varphi_{mn}\rangle}{\langle \varphi_{mn}, \varphi_{mn}\rangle}\,, \qquad m = 0,1,\ldots,s, \qquad n = 0,1,\ldots,t. \tag{5}$$

Thus, solving the problem (1) is reduced to constructing the system of orthogonal polynomials on $\mathcal{P}$ and determining the Fourier coefficients.

Let's write out (5) in more detail:

$$a_{mn} = \frac{1}{\|\varphi_{mn}\|^2}\sum_{i=0}^{M}\sum_{j=0}^{N}\varphi_{mn}(i,j)\,f(i,j) =$$

$$= \frac{1}{\|H_m\|^2}\sum_{i=0}^{M} H_m(i)\left[\frac{1}{\|V_n\|^2}\sum_{j=0}^{N} V_n(j)\,f(i,j)\right].$$

Here $\|H_m\|$ and $\|V_n\|$ denote the norms in spaces $l^2(0:M)$ and $l^2(0:N)$, respectively. Designating by $F_n(i)$ the expression in square brackets, we obtain

$$a_{mn} = \frac{1}{\|H_m\|^2}\sum_{i=0}^{M} H_m(i)\,F_n(i)\,. \tag{6}$$

It is known from the general theory [3] that a system of orthogonal polynomials is unique to within a normalization. We shall use the orthogonal polynomials of the following kind [4]:

$$H_m(x) = \sum_{k=0}^{m}\frac{(-1)^k}{M^{[k]}}\binom{m}{k}\binom{m+k}{k}x^{[k]}\,, \tag{7}$$

$$V_n(y) = \sum_{l=0}^{n}\frac{(-1)^l}{N^{[l]}}\binom{n}{l}\binom{n+l}{l}y^{[l]}\,. \tag{8}$$

Here $x^{[k]} = x(x-1)\cdots(x-k+1)$ is *a factorial polynomial* of order k (by a definition, $x^{[0]} = 1$). The polynomials (7), (8) are normalized by the condition $H_m(0) = V_n(0) = 1$. When such normalization is chosen, the computation of a_{mn} can be effectively performed using recurrent relations [5]. Namely, the following algorithm for solving the problem (1) is offered.

<u>ALGORITHM.</u>

1. Represent $F_n(i)$ as

$$F_n(i) = \sum_{j=0}^{N-n}\psi_n(j)\,\Delta^n g_i(j)\,, \tag{9}$$

where $g_i(j)$ is a trace of function f on the i-th row of the rectangle $\mathcal{P}$, and Δ^n is a finite difference of order n. Coefficients $\psi_n(j)$ satisfy to the recurrent relation:

$$\begin{gathered}
\psi_n(j) = \frac{(j+n)(N+1-j-n)}{j\,(N+1-j)}\,\psi_n(j-1)\,, \quad j=1,2,\dots,N-n, \\
\psi_n(0) = (-1)^n\,\frac{2n+1}{N+1}\,\frac{\dbinom{N}{n}}{\dbinom{N+n+1}{n}}\,.
\end{gathered} \tag{10}$$

Coefficients $\psi_n(0)$ are also calculated recursively:

$$\begin{gathered}
\psi_n(0) = -\,\frac{(2n+1)(N+1-n)}{(2n-1)(N+1+n)}\,\psi_{n-1}(0)\,, \quad n=1,2,\dots,t, \\
\psi_0(0) = \frac{1}{N+1}\,.
\end{gathered} \tag{11}$$

2. Represent a_{mn} as

$$a_{mn} = \sum_{i=0}^{M-m} \eta_m(i)\,\Delta^m F_n(i)\,. \tag{12}$$

Coefficients $\eta_m(i)$ are calculated by the recurrence formulae:

$$\begin{gathered}
\eta_m(i) = \frac{(i+m)(M+1-i-m)}{i\,(M+1-i)}\,\eta_m(i-1)\,, \quad i=1,2,\dots,M-m, \\
\eta_m(0) = -\,\frac{(2m+1)(M+1-m)}{(2m-1)(M+1+m)}\,\eta_{m-1}(0)\,, \quad m=1,2,\dots,s, \\
\eta_0(0) = \frac{1}{M+1}\,.
\end{gathered} \tag{13}$$

A separate computing problem is to determine values of the best mean square approximation polynomial (4). It is possible to use recurrent motives for that, too [6].

Rearranging the addends in the formula (4), represent $\pi_{st}(x,y)$ as

$$\pi_{st}(x,y) = \sum_{m=0}^{s} H_m(x)\left[\sum_{n=0}^{t} a_{mn}\,V_n(y)\right] = \sum_{m=0}^{s} c_m(y)\,H_m(x)\,. \tag{14}$$

Here $c_m(y)$ denotes the sum in the square brackets. For determining values of the polynomial $\pi_{st}(x,y)$ in form (14) we shall use the recurrent relations [4] which the polynomials (7) and (8) satisfy to.

ALGORITHM.

1. Introduce *a Clenshaw sequence* $\left\{b_n^{(m)}\right\}_{n=0}^{t+2}$ as follows:

$$b_{t+2}^{(m)} = b_{t+1}^{(m)} = 0\,,$$

$$b_n^{(m)} - \frac{2n+1}{(n+1)(N-n)}\,(N-2y)\,b_{n+1}^{(m)} + \frac{(n+1)(N+n+2)}{(n+2)(N-n-1)}\,b_{n+2}^{(m)} = a_{mn}\,, \quad n = t, t-1, \ldots, 0. \tag{15}$$

Then for $c_m(y)$ will take place [5]

$$c_m(y) = b_0^{(m)}\,. \tag{16}$$

2. After all coefficients $c_m(y)$ are calculated, construct a Clenshaw sequence for the representation (14):

$$\beta_{s+2} = \beta_{s+1} = 0\,,$$

$$\beta_m - \frac{2m+1}{(m+1)(M-m)}(M-2x)\beta_{m+1} + \frac{(m+1)(M+m+2)}{(m+2)(M-m-1)}\beta_{m+2} = c_m(y)\,, \quad m = s, s-1, \ldots, 0. \tag{17}$$

The zeroth member of the sequence (17) gives the value of π_{st} in point (x, y):

$$\pi_{st}(x, y) = \beta_0\,. \tag{18}$$

In order to find out the effectiveness of the offered algorithms, the numerical experiments of the following kind were carried out. As an initial function we took the polynomial

$$f_{st}(x, y) = \sum_{k=0}^{s}\sum_{l=0}^{t} \xi_{kl}\, x^k\, y^l\,, \tag{19}$$

where ξ_{kl} are independent random values, uniformly distributed on segment $[-1, 1]$. On function (19) the approximating polynomial $\pi_{st}(x, y)$ was constructed. Theoretically, these polynomials should coincide. However because of boundedness of a mantissa and round-off errors, it did not happen. Just the magnitude of the relative error

$$\varepsilon = \frac{\|\pi_{st} - f_{st}\|}{\|f_{st}\|} \tag{20}$$

served as the criterion of effectiveness.

1000 experiments of described kind were carried out on the rectangle $\mathcal{P}$ with $M = N = 100$ and $s, t = 1, 2, \ldots, 10$. The results are assembled in the following table (where $\bar{\varepsilon}$ stands for the average magnitude of the relative error, and τ stands for the average time (in seconds) needed to complete one experiment).

	$s=t=1$	$s=t=2$	$s=t=3$	$s=t=4$	$s=t=5$
$\bar{\varepsilon}$	$2.068 \cdot 10^{-16}$	$1.873 \cdot 10^{-16}$	$2.495 \cdot 10^{-16}$	$2.638 \cdot 10^{-16}$	$2.791 \cdot 10^{-16}$
τ	0.09	0.17	0.28	0.41	0.57

	$s=t=6$	$s=t=7$	$s=t=8$	$s=t=9$	$s=t=10$
$\bar{\varepsilon}$	$2.937 \cdot 10^{-16}$	$3.107 \cdot 10^{-16}$	$3.229 \cdot 10^{-16}$	$3.486 \cdot 10^{-16}$	$3.638 \cdot 10^{-16}$
τ	0.76	0.97	1.20	1.47	1.76

References

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