STANISLAV ALEKSANDROVICH VINOGRADOV, HIS LIFE AND MATHEMATICS

V.P.Havin, N.K.Nikolski

LIFE

S.A.Vinogradov was born in Leningrad on May 1, 1941, on the eve of the German invasion.The siege began when he was a baby of 5 months, his father Aleksandr Antonovich fell at the front, and his mother Olga Ivanovna had to overcome undescribable hardships to save the child from starvation amidst artillery and aircraft bombardments. Postwar years were also hard; her income was scarce (she was a janitor). But she had luck with her son. People who knew him in the forties remember his early addiction to books (mainly classical Russian literature). He was successful at school, especially in mathematics. He joined the mathematical circle of the Palace of Young Pioneers where many Leningrad schoolchildren developed their talents under the guidance of university teachers and students. In his last high school year Stanislav was uncertain about which of the numerous Leningrad institutes to choose for his further education. His hesitation was over after his visit to the mathematics department of the Leningrad State University where G.M. Fichtenholz, a famous professor and brilliant lecturer, gave a talk for high school students (an elementary survey of Calculus). Vinogradov stated his decision as follows: “This is the right place for me to choose, since such people teach here”.

He passed his entrance exams quite successfully and impressed his examiners. Slava quickly became one of the brightest students. The first author of this preface had him in his class and remembers Vinogradov’s vivid and keen interest in the subject (power series in the spirit of Landau’s “Darstellung und Begründung”, the Hardy classes, and other spaces of analytic functions). His reaction was always sharp and swift. G.P. Akilov who was giving the course on Functional Analysis was impressed by the following episode. In one of his introductory lectures he proved that the intersection of any decreasing sequence of closed balls in a complete metric space is non-empty if the radii tend to zero; he had then mentioned that (maybe) the last condition is redundant. Vinogradov protested immediately, saying that this is so in a Banach space, but not in general.

The following reminiscence illustrates Slava’s addiction to mathematics in his early student years. In the late 50’s and the early 60’s, every summer and fall, the university students were used as an unpaid agricultural labour force, picking potatos, cabbages, etc.

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Now, please try to imagine the surrealistic picture to which the second author, being Slava's closest university classmate, was a witness: a dark, cold and wet fall of 1959 in a kolkhoz lost amidst the impassability of the northern Leningrad region; a big room in a barrack where 25 guys sit on plank beds playing cards, shouting, smoking, drinking wine... In the center of the room, under the only lamp, a sophomore, S.Vinogradov, is sitting cracking problems from the Polya and Szegö Problem Book, and, when tired of Polya and Szegö, studying a French memoir by R. Baire "Sur les fonctions de variables réelles", Ann. Mat. Pura e Appl. (3), 3(1899), 1-122...

His talent was clear to us since his student years (the first author was his adviser, the second was his fellow student and closest friend). He was a born analyst, the one "with a square skull" as the saying goes. He liked abstract "soft" approaches, but his most natural inclination was an elegant "hard" estimate.

Vinogradov graduated from the university in 1963, being already the author of nice results on the (fashionable at the time) free interpolation phenomena (see below); one of these results was mentioned by J.-P. Kahane who included it in his book on absolutely convergent Fourier series.

Vinogradov's living conditions during this period were still quite painful. The second author remembers a rather anecdotal but colourful episode from the First Summer Analysis School in Uzhgorod (Carpathian mountains, 1964): invited to an official conference reception held in a high-standing restaurant, Stanislav shocked the waiters by ordering... five milk buckwheat puddings one after another, instead of the proposed delicacies. In fact, this was an ineffaceable relic of siege starvation, the well-known phenomenon among the siege survivors who subconsciously harbored forever a fear of permanent hunger. At about the same time, Stanislav, already a PhD student at Leningrad University, was forced to take a second job, spending hours and hours at a noisy electrical "Reinmetall" calculator and doing long computations for an applied project directed by Prof. L.M. Kachanov. (By the way, a result of this involvement was that he found a mistake in an engineering algorithm, and that had far-reaching consequences!). Even much later, when already a university professor, he did not have his own desk for a long time and often worked locked in the bathroom of his small apartment, the only place where he could isolate himself for a while.

A brief and formal description of his further career follows: in 1963 he became a graduate student at Leningrad university; in 1966 he was appointed an assistant professor; his first thesis was defended in 1968; he became an associate professor in 1969 and in 1983 defended his second thesis. In 1987 he got his full professorship and the Chair of Analysis at Leningrad University.

He was an enthusiastic and conscientious teacher. Teaching, and methods to present mathematics to beginners in an optimal way combining clarity and brevity were his constant preoccupations, the theme of innumerable conversations and disputes with colleagues with whom he generously shared his pedagogical findings. Being very inventive he often contrived to find clever tricks, simplifying and perfecting even the most common points in standard courses, not to mention more intricate topics.

As to PhD students, he had few of them formally speaking (chronologically ordered, A. Shtern, A. Kotochigov, M. Bludze, T. Andrianova, S. Shimorin, P. Stoilov, A. Petrov).

But instead he continually and informally worked with half of the huge Leningrad Analysis seminar. Being a very enthusiastic solver of tooth-breaking puzzles, S.V. was easily captivated by somebody else's problems. Once involved, he often inspired colleagues with an idea leading them out of a dead end. As a result, many of Vinogradov's important achievements are scattered throughout his friends' published papers. Without trying to be exhaustive we give a list of such results at the end of this Preface.

His duties were varied, numerous, and not always pleasant. The most pleasant one (besides teaching) was the supervision of budding analysts, i.e. of their Master's and PhD theses. But his duties included some purely administrative tasks, elements of routine, red tape, etc. He accepted all that as an inevitable burden, his duty to his comrades and the department. His conduct and the decisions he had to take were invariably marked by deep innate decency. To illustrate this we describe a typically Soviet moral dilemma he had to confront after he became the Chairman of the Analysis Department. For a leader of this rank to act efficiently, to be in a position to protect his colleagues from (not always friendly)bosses or to take a serious part in decision making at the departmental level it was an absolute necessity to be a Party member. Many friends (including-alas! - the first author) put strong psychological pressure on him to convince him to comply with this imperative and forget his scruples in favour of the common interest. He agreed that these considerations were serious, but was extremely reluctant to fulfil what everybody wanted from him. He was not a political dissident, a fervent adversary of the regime, not at all. Having benefited fully from the advantages of the Soviet educational system which ensured his access to a high level of the scientific hierarchy, and having been brought up in communist ideology, he respected and accepted the old noble slogans like internationalism, social justice, etc. But at the same time he hated the stupid and hideous sides of everyday Soviet life. He knew that a Party member couldn't escape taking part (be it passively) in some disgraceful actions (or, at least, in their formal approval). This is why, when attacked by somebody insisting once again that his Party membership was a must, he would nod, but not accelerate the process. He lingered thus for approximately five years, postponing and promising, until perestroika and subsequent events solved the problem by just removing it. But apart from similar nuisances he was lucky–with his friends, his career, his family. It is a pleasure to remember merry May day parties (his birthday) at a picturesque table laid by Irina, Slava's wife, with his son Aleksandr, in the company of his friends...

Stanislav was a very balanced personality, kind of a practical philosopher. In professional life, he always ignored any superficial "pseudo mathematical" fuss: elections, honours, titles. He disliked trying to push himself into "prestigious" mathematical journals, where, to please varying mathematical fads, or an "ideological" point of view, or simply to serve some group interests, various protagonists sometimes try to teach you how and what should be done in Analysis. (The majority of Vinogradov's beautiful papers were published in the purely domestic "Zapiski Nauchnyh Seminarov LOMI (Steklov Institut of Mathematics, Leningrad)"). Instead, S.V. was confident of the intrinsic value of hard analysis, no matter which fashion was imposing its whims outdoors.

His insidious illness progressed slowly, inexorably. In the summer of 1997, speaking to one of us about his still continuing mathematical plans, he would add repeatedly, as a refrain, "if I have enough time..." (although, formally speaking, the diagnosis was kept from him, due to Russian medical practice). His last months were hard, he struggled against excruciating pain. Starting in the early fall of 1997 he was physically incapable of lecturing at the university, but still prepared his lectures and thoroughly instructed the younger colleagues who replaced him. He would joke: "I am a home-lecturer"... The first author will never forget his last mathematical phone conversation with Stanislav (it was about the de Branges space $H(E)$'s being isometric to $H^2 \ominus IH^2$ for an appropriate inner function I; Slava promptly proved the conjecture by an explicit construction connecting E and I...). He was an unforgettable example of spiritual strength and courage.

Many will always miss Stanislav, a good friend, ready to help, clever (even wise, which quality he demonstrated more than once when supporting and advising his friends in delicate situations). He was a bright personality with a vast field of cultural interests. An admirer of the magnificent Russian language of N.Gogol's prose and a passionate music lover, he did not miss any interesting concerts in the Leningrad Philharmonia Main Hall. In his younger years, he was there almost every week: gallery tickets cost less than a modest lunch. S.A. was a deep connoisseur of classical vocal music and, having himself a beautiful baritone, he often sang in friendly gatherings with wine...

But mathematics was above all.

Mathematics

To locate Vinogradov's interests on the map of mathematics we start with a well-known theme of classical analysis discussed with a great force of expression in an old article [L] by Paul Lévy under a striking and somewhat emotional title. It is devoted to some "unsolved and surely unsolvable" problems of Fourier analysis. P.Lévy mentions, among many other things, the problem of finding a non-tautological and palpable description of the sequences $c = (c_n)_{n\in\mathbb{Z}}$ of complex numbers such that

$$c_n = \hat{f}(n), \qquad n \in \mathbb{Z} \tag{1}$$

for some function $f \in C(\mathbb{T})$, the space of all functions continuous on the unit circle $\mathbb{T}$; here $\hat{f}(n)$ is the n-th Fourier coefficient of f,

$$\hat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} dm(z), \tag{2}$$

and m is normalized Lebesgue measure on $\mathbb{T}$. The meaning of the adjectives "palpable" and "non-tautological" is illustrated by the Riesz-Fischer description of sequences c satisfying (1) with $f \in L^2(\mathbb{T})$ in place of $C(\mathbb{T})$. And any practicing classical analyst can add a lot of his own examples of such "sure non-solvability".

Problem (1) evoked by P.Lévy is a trigonometrical moment problem, representative of the large family of "moment problems" including the Hausdorff, Stieltjes, and Hamburger "power" moment problems

$$c_n = \int_{\mathbb{R}} x^n d\mu(x) \qquad (n \in \mathbb{Z}_+), \tag{3}$$

corresponding to various classes of real measures μ on $\mathbb{R}$.

This subject is important and rich in impressive connections, ideas, applications, and deep results. However, looking at them leaves (as a rule) an aftertaste of dissatisfaction. If, in order to decide whether a concrete sequence c satisfies (3) or (1), you are invited to test the positiveness of an infinite family of quadratic forms (or to use another similar procedure), you feel a bit disappointed. But experience shows that nothing more feasible can be proposed, and the feeling of "unsolvability" persists.

The same uneasiness is often caused by the interpolation problems of complex analysis. If you need a description of traces of a class of analytic functions on a given set, be ready to get something indigestible.

These phenomena of "sure non-solvability" often stem from prohibitions imposed by the Uncertainty Principle of harmonic analysis, which makes the simultaneous command and coordination of "time" and "spectral" properties so difficult. Another (but closely related) source lies in the uniqueness properties of analytic functions which entail rigid (and often mysterious) interconnections between the values of a function at different points.

But usually there is a way to modify the setting of a "surely unsolvable" problem so as to make the solution quite satisfactory and transparent, with all obscure constraints disappearing as if by miracle. Turn once again to Lévy's example (1). A necessary condition of solvability is obvious:

$$\sum |c_n|^2 < +\infty. \tag{4}$$

It is very far from sufficient, and nothing palpable can be done beyond it. But if we moderate our appetites and replace (1) by

$$c_n = \hat{f}(n), \qquad n \in E \tag{5}$$

with a *sparse* (infinite) $E \in \mathbb{Z}$, then (4) becomes sufficient for the solvability of (5)! The "truncated" trigonometrical moment problem (5) is *freely* solvable.

Such phenomena of liberation, involving the free solvability of an initially overdetermined problem, were central for Vinogradov and stimulated his research in many other directions.

Another important idea illuminating many of his results is the duality between bases and interpolation problems, or more generally, between the geometry of reproducing kernels for a given holomorphic space and the "freedom" of its traces on sparse subsets. He paid a lot of attention to this duality (see below), and his impact on its development was considerable. The third feature of his mathematical style is his belief in the tools of real analysis, in a kind of positive operator inequalities underlying holomorphic structures.

Our article is followed by the translation (by V. Havin) of the prefaces to Vinogradov's theses ([P1], [P2]). They form a good survey of large parts of his work, with motivation and historical background. We complement [P1], [P2] turning to results not covered by these texts.

Free interpolation. The meaning of this term has been partly explained in the preceding remarks, see also [P1], [P2]; it often means that the restriction space $X|\Lambda$ on a given set Λ is a lattice, see [P1], [P2]. Vinogradov's contribution to this theme includes, besides numerous new results, a less spectacular but maybe not less important point, namely, a proof of Carleson's free interpolation theorem for H^∞ based on the so-called "Vinogradov-Senichkin test" (see the next section devoted to embedding theorems). This proof, devised by Vinogradov in 1974, is technically simpler and shorter than the original one. Moreover, it gives the Carleson interpolation condition a new form leading to a precise two-sided estimate of the interpolating function. The qualitative and quantitative advantages of Vinogradov's proof as well as a shorter and quite elementary proof by P. Jones are discussed in detail in Appendix 1 to the monograph [Koo], the proof can be also found in [Nik] and [12] (numbers in square brackets refer to Vinogradov's list of publications).

In [29] free interpolation with multiplicities in H^∞ is studied. New proofs and generalizations of the Nikolskii-Vasyunin results are given there. Vinogradov's approach works perfectly in this more difficult situation too (including the most interesting case of unbounded multiplicities), where the Jones explicit interpolation formulas are impossible (as is proved in [29]). In particular, the H^p free interpolation of locally defined germs of the spaces $H^p(L^\epsilon)$ on the level sets L^ϵ is considered, where $L^\epsilon = \bigcup_{\lambda\in\Lambda}\{|b_\lambda(z)|^{k_\lambda} < \epsilon\}$ and the Blaschke product $B(z) = \prod_{\lambda\in\Lambda} b_\lambda(z)^{k_\lambda}$ satisfies the generalized Carleson condition. It is proved that to be "free", the interpolation cannot be ensured for sets smaller than L^ϵ, $0 < \epsilon < 1$.

Some Jones-like formulas are constructed in [30]. They yield a lot of new information on interpolating functions. In particular, any bounded function defined on a Carleson set $E \subset \mathbb{D}$ (= open unit disk) can be interpolated by a function in $H^\infty \cap K_{B^2}$, where $K_{B^2} := H^2 \ominus B^2H^2$, and B is the Blaschke product vanishing precisely on E. An intersting new feature of free interpolation phenomena was discovered in [14]. This paper, also written in the spirit of the bases-interpolation duality, is commented on in the preface to Vinogradov's second thesis [P2]. However, here we would like to stress that it contains the very first result showing that even for usual Banach spaces $X \subset Hol(\mathbb{D})$ the "freely interpolated" lattice-type restrictions $X|\Lambda$, $\Lambda \subset \mathbb{D}$ can be completely different for different Λ's, and not even isomorphic as Banach spaces. In [14] this is done for the spaces

$$X = l^p_A = \{f \in Hol(\mathbb{D})) : f(z) = \sum_{n\geq 0} \hat{f}(n)z^n, \sum_{n\geq 0} |\hat{f}(n)|^p < \infty\}, \ p \neq 2$$

and their half plane analogs $\mathcal{F}L^p(0,\infty)$. (By the way, being a concrete analyst S.V. was very far from the snobbish opinions to the effect that "true" analysis is limited to the Lebesgue norms of L^2, L^1, and L^∞. He seriously contributed to L^p spaces, and, as A.Markus recalled to one of us, polemically said in one of many friendly discussions: "Let them write "$p \neq 2$" on my tomb".)

Vinogradov made some ingenious observations on bases of exponentials and, more generally, on reproducing kernels for the model spaces K_Θ. Namely, he found the first example of a 'sine-type function' (in B.Ya. Levin's sense) with zeros not lying in a strip, and gave the first sufficient conditions for a family of exponentials $(e^{i\lambda t})_{\lambda\in\Lambda}$ to admit completion to a Riesz basis of $L^2(0,a)$, $a > 0$. See the last Section of this Preface for references and more details.

A famous example of free interpolation is the Rudin-Carleson theorem: if $E \subset \mathbb{T}$ is closed and $m(E) = 0$ then any $f \in C(E)$ coincides on E with a function from the disc-algebra

$C_A(\mathbb{D})$, the set of functions continuous on $\mathbb{D}\cup\mathbb{T}$ and analytic in $\mathbb{D}$. In [21] Vingradov and Hruščev turn to similar phenomena in U_A,

$$U_A := \left\{ f \in C_A : \sum_{n\geqslant 0} \frac{f^{(n)}(0)}{n!} z^n \text{ converges uniformly in } \mathbb{D} \right\}.$$

One of their main tools is the far-reaching Vinogradov generalization of the Kolmogorov inequality for the Hilbert transform (see 0.9 in [P2]). The starting point of [21] is the following stronger version of the Rudin-Carleson theorem due to Oberlin: $U_A|E = C(E)$ if $E \subset \mathbb{T}$ is closed and $m(E) = 0$. The authors consider, besides E, a set $\Lambda \subset \mathbb{Z}_+$ and the operator

$$f \mapsto \left(f|E, \hat{f}|\Lambda\right) \in C(E) \times l^2(\Lambda),\ f \in U_A,\ \hat{f} := \left(f^{(k)}(0)/k!\right)_{k\in\mathbb{Z}_+}.$$

They prove its surjectivity provided that $m(E) = 0$ and that Λ satisfies the $\Lambda(2)$-condition (i.e. $||p||_{L^1(\mathbb{T})} \asymp ||p||_{L^2(\mathbb{T})}$ for any trigonometrical polynomial p). Thus, freedom of interpolation is double: we are free to interpolate the values prescribed on E and, simultaneously, the Maclaurin coefficients with indices in Λ.

Another result of [21] is the existence of a linear interpolating operator in the Oberlin theorem. Functions in H^∞ with uniformly bounded partial sums of their Maclaurin series are also studied (together with the multipliers of this class).

This article contains an extremely useful result on the Hilbert transform $\tilde{\mu}$ of a complex Borel measure μ on $\mathbb{T}$:

$$\tilde{\mu}(z) := p.v. \int_{\mathbb{T}} \frac{d\mu(t)}{t - z}$$

Theorem.

$$\lim_{y\to+\infty} y\, m\bigl(\{z \in \mathbb{T} : |\tilde{\mu}(z)| > y\}\bigr) = \frac{var\ \mu_s}{\pi};$$

where μ_s denotes the m-singular part of μ.

This theorem was applied to the study of multiplicative properties of Cauchy-type integrals in [Gol], and to that of the Cauchy problem for the Laplace equation in [ABG].

Vinogradov's influence was important for research on the free interpolation of analytic functions smooth (in various senses) up to the boundary. One of the initial results in this direction is due to his student A.Kotochigov [Kot], who considered free interpolation on $\mathbb{T}$ and in $\mathbb{D}$ of functions analytic in $\mathbb{D}$ and satisfying a Lipschitz condition or belong to a Sobolev space. These results stimulated works by Ye.Dyn'kin [Dy1], [Dy2] who found a complete description of sets of free interpolation. In [23] free interpolation in some Besov classes of analytic functions is considered. Free interpolation in the spaces of Cauchy type integrals and of their multipliers is one of the themes of [13].

To conclude this section we mention [28], where free interpolation is used as a tool to improve known estimates of $\|P_+ f\|_\infty$ for a function f meromorphic in $\mathbb{D}$ having at most n poles and with $\|f\|_\infty \leq 1$; P_+ being the Riesz projection.

Carleson-type embedding theorems. To understand properties of a function space X it is often advisable to look at an "accompanying" space, like, say, the dual space X'. Another example is the algebra mult$X$ of all multipliers of X, one of Vinogradov's favourites (see below). One more example is the set $\mathrm{carl}_p X$ of all positive measures μ such that $\int_E |f|^p \, d\mu < +\infty$ for any $f \in X$ (E is the domain of functions in X). If X is a Banach space, then, as a rule, $\mu \in \mathrm{carl}_p X$ satisfies

$$\|f\|_{L^p(\mu)} \leqslant c(\mu)\|f\|_X, \quad f \in X,$$

so that X is continuously embedded in $L^p(\mu)$ whence the term "an embedding theorem". A classical sample is the Carleson embedding theorem yielding a very clear and efficient description of $\mathrm{carl}_p H^p$ (where $H^p = H^p(\mathbb{D})$ is the usual Hardy space). This result is crucial for free interpolation in H^p (including for $p = +\infty$). Its initial proof, based on some delicate "dyadic" techniques, wasn't easy. V.Senichkin simplified it essentially [Sen], and then Vinogradov turned it into something *quite* simple having reduced it to what is now known as the Vingradov-Senichkin test ([Nik], lecture VIII). This is an easy estimate of the norm of an integral operator with a positive kernel acting in $L^2(\mu)$, μ being an abstract measure; its most sophisticated ingredients are the Cauchy-Schwarz inequality and Fubini's theorem. The test or its elements had appeared in the past here and there in implicit form (e.g., in a manuscript of Luzin published in [Luz], p. 287–319, or in theorems on the quasi- or almost everywhere convergence of Fourier series whose coefficients belong to a weighted l^2-space [Carl, ch. V]); it is a tool ready to be used on many occasions. An impressive application is the proof of a new version of the Carleson embedding theorem. The description of $\mathrm{carl}_2 H^2$ appears in a disguised, seemingly implicit (but equivalent) form. At first glance this proof is not as clear geometrically as Carleson's, but it turns out to be much more practical (see the discussions in [Nik], Lecture VIII, and in [Koo], Appendix 1). The generality and power of this approach to "L^2-embeddings" are demonstrated by Vinogradov's axiomatic schemes and further results on $\mathrm{carl}_2 X$ for many Hilbert spaces X of analytic functions ([P2], [31], [34]). Some of his ideas and results related to this scheme are explained and discussed in [MSh], §3.11; their influence is acknowledged in [Ver1], [Ver2], [VerK], and [TV]. The latter paper also contains impressive applications of the test to matrix valued Hunt-Muckenhoupt-Wheeden theorems, as well as interesting modifications and improvements of the test itself giving, in particular, the simultaneous continuity of two operators with different kernels; see theorems 7.3 and 7.4 there.

From a more general point of view, we can say that the Vinogradov-Senichkin test is one of the first and most explicit manifestations of what we can call the Reproducing Kernel Thesis (RKT). The RKT says, "an operator $T : H \longrightarrow H$ defined on a Hilbert space $H \subset Hol(\mathbb{D})$ with the reproducing kernel k_z, $z \in \mathbb{D}$, is bounded if it is bounded on reproducing kernels, that is, if $\sup_{z\in\mathbb{D}} \|Tk_z\|/\|k_z\| < \infty$." Of course, without any hypothesis on a linear operator T, the thesis fails. But RKT works for many important operators related to analysis on H ; why and under which assumptions, is still a mystery. The Vinogradov-Senichkin test proves RKT for embedding operators $H^2 \subset L^2(\mu)$, and in fact gives more. The RKT is true for Toeplitz operators $Tf = T_\varphi f = P_+\varphi f$ on H^2, and the Bonsall-Power theorem [BP] says that it is also true for Hankel operators $Tf = H_\varphi f = P_-\varphi f$ on H^2 (here P_+ means the Riesz orthogonal projection onto H^2, and

$P_- = I - P_+$). This is equivalent to saying that the known Garsia norm is equivalent to the usual BMO norm. R. Coifman, D. Luecking, R. Rochberg, K. Zhu, and many others have proved the RKT for Toeplitz and Hankel operators on many other spaces with reproducing kernels, including some spaces of functions of several variables (Bergman, Dirichlet, etc); see [Zhu] for results until 1990. The Axler- Chang- Sarason- Volberg theorem (see [Nik], for example) says that the RKT is true for semi-commutators of Toeplitz operators. The famous Muckenhoupt (A_2) condition is nothing but the RKT for the Riesz projection P_+ on a weighted space $L^2(\mathbb{T}, w\,dm)$, and, as is shown in [TV], the only way to describe matrix-valued Muckenhoupt weights is to use the RKT. We end this list by referring to the theory of the Berezin transform $z \longmapsto (Tk_z, k_z)$ (see [Ax], [Zhu]) which is closely related to the RKT, and pointing out once again that Vinogradov's approach to embedding theorems is still one of the most useful examples of the RKT.

A generalization of the Carleson embedding theorem is obtained in [39]; it is applicable to spaces A_p^1 (to be defined at the end of the next section).

Multipliers of spaces of analytic functions. Let X be a class of functions analytic in $\mathbb{D}$. A multiplier of X is by definition a function f analytic in $\mathbb{D}$ such that $f \cdot x \in X$ for any $x \in X$; the set of all multipliers of X is denoted by $\operatorname{mult} X$. Vinogradov studied mainly $\operatorname{mult} \ell_A^p$ and $\operatorname{mult} \mathcal{K}_A$ where ℓ_A^p is the space of functions $\sum\limits_{n\geq 0} c_n z^n$ with $\sum\limits_{n\geq 0} |c_n|^p < +\infty$, and $\mathcal{K}_A$ is the space of Cauchy-type integrals $\int\limits_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}$ (in $\mathbb{D}$), formed for complex Borel measures μ on $\mathbb{T}$.

Results for the former class are described in [P2]. Here we only point out the short problem notes [33], [37], and briefly discuss the elegant results of [39], the last item in Vinogradov's list of publications.

Let V be a Banach space contained in $L^\infty(E)$ where E is a set equipped with a measure. Suppose $\|f\|_\infty \leq \|f\|_V$ for any $f \in V$. A function Φ analytic in a disc $\{z : |z - c| < r\}$ is said to act on V if

$$\{\Phi \circ f : f \in V,\ \|f - c\|_V < r\} \subset V.$$

Any such Φ acts on V provided that V is a Banach algebra with respect to pointwise multiplication. The authors of [39] proclaim the following dichotomy: either V is a Banach algebra or Φ does not act on V unless it is affine (i.e., unless $\Phi'' \equiv 0$). This is not a formal assertion but rather a useful heuristic principle confirmed by some precise statements and many concrete examples. Among other things a question asked in [Nik2] is answered in [39]: if $1 < p < 2$, then $1/f$ may not belong to ℓ_A^p for an $f \in \ell_A^p$ separated from zero (i.e. $\inf\limits_{\mathbb{D}} |f| > 0$). The above principle proves to be applicable to $V = U_A$ and $V = \ell_A^p \cap C_A$ ($1 < p < 2$). The last example entails the following corollary: $\ell_A^p \cap C_A$ is not an algebra if $1 < p < 2$, an unpublished result that J.Bourgain obtained by a completely different method. Many similar and much subtler results are proved in [39]. It contains delicate norm estimates in $\operatorname{mult} X$ for various spaces X based on random power series techniques.

The properties of $\operatorname{mult} \mathcal{K}_A$ turned out to be rather mysterious. It is a proper subset of H^∞; its elements possess some regularity properties (radial boundary values exist everywhere on $\mathbb{T}$, partial sums of the Maclaurin series are uniformly bounded), but none of these can characterize the functions in $\operatorname{mult} \mathcal{K}_A$. First results obtained in [H1], [H2], [7] led to the following question: which inner functions are in $\operatorname{mult}\mathcal{K}_A$? (A function $I \in H^\infty$ is called inner if $\lim\limits_{r\to 1} |I(r\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$.) It was answered in [22]. To state this beautiful solution we need the following definition: a sequence $(a_n)_{n=1}^\infty$ of points in $\mathbb{D}$ is called a Frostman sequence if

$$\sup\left\{\sum_{n=1}^{\infty}\frac{1-|a_n|^2}{|\zeta-a_n|} : \zeta \in \mathbb{T}\right\} < +\infty.$$

Theorem [22]. *Let I be an inner function. The following are equivalent: (1) $I \in \operatorname{mult}\mathcal{K}_A$; (2) I is a Blaschke product whose zeros form a Frostman sequence.*

The proof is based (in part) on the results of [7] and [13], but contains new ingredients including free interpolation techniques (the problem itself does not mention any interpolation). As a byproduct [22] offers a nice construction of a Blaschke product converging everywhere on $\mathbb{T}$ with a prescribed closed nowhere dense set of points of discontinuity on $\mathbb{T}$, its Maclaurin series converging everywhere on $\mathbb{T}$ and the n-th Maclaurin coefficient being $O(\frac{1}{n})$. Another result in [22] is a complete description of the families $\left(\frac{1-|\lambda|^2}{1-\overline{\lambda}z}\right)_{\lambda\in\Lambda}$ forming an unconditional basis for their closed linear span in $\operatorname{mult}\mathcal{K}_A$. The following lemma of [22] sheds new light on the class $\operatorname{mult}\mathcal{K}_A$: *a function $\varphi \in H^\infty$ is in $\operatorname{mult}\mathcal{K}_A$ iff the Toeplitz operator $T_{\overline{\varphi}}$ acts continuously in the disc-algebra C_A ($T_{\overline{\varphi}}(f) := P_+(f\overline{\varphi})$, $f \in C_A$, where P_+ is the M.Riesz projector: $P_+(p) = \sum\limits_{n\geq 0} c_n z^n$ for trigonometrical polynomials $p = \sum\limits_{n\in\mathbb{Z}} c_n z^n$).*

More generally, multipliers play an important role in the theory of Toeplitz operators. In particular, Vinogradov's results on the spaces mult l^p_A, partly presented in [P2], were essentially used in the exhaustive monograph [BS], Ch. 6. We mean first of all the results from [11], [19] on the membership problem (sufficient conditions), and also the divisibility property of mult l^p_A described in the next Section.

Let us conclude our discussion of multipliers by turning to the space A^1_p of functions f analytic in $\mathbb{D}$ and satisfying

$$\int\limits_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-1}dxdy < +\infty,\ z = x+iy,\ 0 < p \leq 2;$$

thus, A^1_1 is the "analytic" Sobolev class $W^1_1(\mathbb{D})$, A^1_2 is the Hardy class $H^2(\mathbb{D})$, and $\ell^p_A \subset A^1_p \subset H^p(\mathbb{D})$. A function $f \in A^1_p$ can also be looked on as the Cauchy transform of a function in a Besov class on $\mathbb{T}$. In [38] a nice and efficient description of $\operatorname{mult} A^1_p$ is obtained; another result is a description of some inner functions in $\operatorname{mult} A^1_p$.

Note that the partial result that $\operatorname{mult}(A^1_1) = H^\infty \bigcap \{f : |f'|\,dxdy$ is a Carleson measure $\} = \{f \in Hol(\mathbb{D}) : \sup_{\lambda\in\mathbb{D}} \|f\circ b_\lambda\|_{A^1_1} < \infty\}$ was obtained by Vinogradov much earlier and was first published in V.Peller's paper [Pe1] (with references to S.V.), where it is applied to estimates of functions of Hankel operators.

Inner-outer factorization (division problems). Any function f of the Smirnov class (in particular, of any Hardy class $H^p(\mathbb{D})$) can be decomposed into the product of two factors:

$$f = I_f \cdot O_f$$

where I_f is inner and O_f is outer (O_f does not vanish in $\mathbb{D}$, and $\log|O_f|$ is the Poisson integral of its boundary values). This inner-outer factorization is unique up to a constant factor and is often called canonical. Its role in complex and harmonic analysis and in various applications is outstanding (see, e.g., [Nik], [Koo]; some historical remarks are in [NikH]). In many cases O_f inherits properties of f. For example, $O_f \in H^p$ or $O_f \in C_A$ whenever $f \in H^p$ or $f \in C_A$ respectively. The following precise and more general question is natural and interesting: given a class X of functions of the Smirnov class and an inner function I, is it true that if $f = I \cdot O \in X$ and O is analytic in $\mathbb{D}$, then $O \in X$? If so, we say that division by I is possible in X.

Vinogradov was one of the first who solved this division problem for interesting pairs X, I (for the present state of this circle of problems see [Sh]).

There is a simple way to prove divisibility by an inner factor I: if the Toeplitz operator T_I acts in $X \subset H^1$, then division by I is possible in X (since $T_I(IQ) = Q$), and the boundedness of T_I can often be obtained for free by duality. This approach yields a lot of divisibility theorems, but it fails in some "extreme" cases due to the unboundedness of the Hilbert transform in a concrete space, and it is just these cases that turn out to be the most interesting. They usually need delicate estimates to show that a certain regularity of f is due to O_f alone, not to a cancellation of singularities of I_f and O_f (I_f is, as a rule, very irregular).

One such subtle situation was the theme of [8] where the divisibility by any inner function was proved for $X = \{f \in H^1 : f' \in H^p\}$, $p = 1$ or $+\infty$ (if $1 < p < +\infty$, then this can be proved by the "soft" Toeplitz-operator method). Actually, [8] gives more than mere divisibility: a precise two-sided estimate of $\|f'\|_{H^1}$ is given in terms of independent parameters defining O_f and I_f ($|f| \mid \mathbb{T}$, the zeros of f, and the singular positive measure figuring in I_f).

The second part of [38] contains positive and negative results on the division problem in A^1_p (see the preceding section for the definition). In particular, division by the elementary singular inner function $I = exp[ai\frac{z+1}{z-1}]$ $(a > 0)$ is impossible in spite of the inclusion $I \in \mathrm{mult}A^1_p$.

In [13] Vinogradov proved that in $X = \mathcal{K}_A$, division by an arbitrary inner factor is possible. A different proof was found later by A.B.Aleksandrov [Al]; interesting refinements are in [Pol]. In [13] a general scheme is proposed to prove the divisibility by an inner factor in a space of analytic functions.

Uniqueness theorems, zero sets. This is the theme of items [1] (discussed in [P1]), [10], and [34].

A set $E \subset \mathbb{D} \cup \mathbb{T}$ is said to be a uniqueness set for a class X of functions analytic in $\mathbb{D}$ if any $f \in X$ vanishing on E vanishes identically; for $\zeta \in \mathbb{T}$ the value $f(\zeta)$ is understood as $\lim\limits_{r \nearrow 1} f(r\zeta)$.

In [10] the uniqueness sets for the class $H^1_1 := \{f \in H^1 : f' \in H^1\}$ are described. The description looks very familiar to anybody who has ever dealt with similar problems: a closed set $E \subset \mathbb{D} \cup \mathbb{T}$ is a uniqueness set for H^1_1 iff

$$\int\limits_{\mathbb{T}} log\rho_E dm = -\infty \tag{6}$$

where $\rho_E(\zeta)$ is the distance from ζ to E. The necessity of (6) follows immediately from the Carleson theorem on uniqueness sets for analytic functions smooth (say, C^∞) up to the boundary. The sufficiency would be quite easy if, in place of H_1^1 we had a class X consisting of Lipschitz functions (of any order): in that case (6) implies the divergence of $\int_{\mathbb{T}} log|f|dm$ for any $f \in X$ vanishing on E. However, for $f \in H_1^1$ the continuity modulus cannot be controlled (although $H_1^1 \subset C_A$), and to deduce uniqueness from (6) is a delicate matter. Actually, in [10] a nice upper estimate on the geometrical mean $G(|f|) = \exp\left(\int_{\mathbb{T}} \log|f|\, dm\right)$ for $f \in H_1^1$ over $\mathbb{T}$ is proved; $G(|f|)$ is majorized by $G(\rho_E|f'|)$.

The article [34] is devoted to the role played by the Blaschke condition

$$\sum_{k=1}^{\infty}(1-|a_k|) < +\infty, \qquad a_k \in \mathbb{D}, \tag{7}$$

in uniqueness theorems. This condition is necessary for the existence of a non-zero function of the Nevanlinna class (not to mention the Hardy classes) vanishing at $a_k,\ k = 1, 2, \ldots$. However, a set $\{a_k\}$ satisfying (7) may well be a uniqueness set for a class X of Hardy functions. For example, this is the case if $X \subset C_A$ (= the disc algebra), and the closure of $\{a_k\}$ contains $\mathbb{T}$, which is compatible with (7). In [34] much subtler situations are considered when X contains functions everywhere discontinuous on $\mathbb{T}$ or even not possessing radial boundary values. A general method of constructing such classes X and uniqueness sets for them satisfying (7) is proposed. The list of classes X considered includes the spaces mult$\mathcal{K}_A^p$, $p = 1, +\infty$, where $\mathcal{K}_A^p$ denotes the set of all Cauchy transforms of functions in $L^p(\mathbb{T})$. Another example is the class of Cauchy transforms of the functions

$$f \in C^\omega(\mathbb{T}) := \{f \in C(\mathbb{T}) : \omega_f \le c_f\omega\},$$

where ω is some given increasing positive function and ω_f denotes the continuity modulus of f. An interesting case occurs when

$$\int_0 \frac{\omega(t)}{t} dt = +\infty$$

(otherwise our class is in C_A). The approach of [34] works for some analytic Besov classes and some classes of power series characterized by the rate of decay of their coefficients. Results of [34] are closely related to the division problems discussed in the preceding section.

Singular integrals. Vinogradov's generalization of the Kolmogorov weak type estimate of the Hilbert transform is stated and discussed in [P2]. Here we only mention some repercussions of this result.

Bourgain applied a modification of Vinogradov's theorem to the study of the space $U(\mathbb{T})$ of uniformly convergent Fourier series (see [B]). In [K1] this result of Vinogradov was used to prove a quantitative refinement of the Men'shov correction theorem: *for any* $\epsilon \in (0, 1)$ *and* $f \in C(\mathbb{T})$ *there is a* $g \in U(\mathbb{T})$ *such that*

$$m(\zeta \in \mathbb{T} : g(\zeta) \ne f(\zeta)) < \epsilon,$$

and $\|g\|_U \leq const \cdot \log \frac{1}{\epsilon}\|f\|_\infty$*; this inequality is sharp.* In [K2] Vinogradov's result is again called on and leads, this time, to a solution of the long-standing Szidon problem for U_A: *given* $(x_n)_{n\in\mathbb{Z}_+} \in l^2$, $x_n \geq 0$, *there is an* $f \in U_A$ *such that* $|f^{(n)}(0)|/n! \geq x_n$, $n = 0, 1, 2, \ldots$. In [K3] Vinogradov-like estimates again emerge in a new context.

As a whole, Vinogradov's generalization of the Kolmogorov estimate and its consequences show that the spaces $U(\mathbb{T})$ and U_A of uniformly convergent Fourier series are much closer to the familiar $C(\mathbb{T})$ and C_A than it can be predicted from preceding experiences of classical analysis (properties of the Dirichlet kernels, the Lebesgue constants, etc.)

We also mention estimates of Cauchy-type integrals on curves obtained in [17]. They may seem obsolete after the immense progress made in this area in the eighties. We believe, however, that the concrete norm inequalities of [17] and their proofs may be of interest even now. For example, let $\Gamma = \omega(\mathbb{T})$ where ω is a diffeomorphism of $\mathbb{T}$ such that the Fourier series of $s \mapsto \frac{d}{ds}\omega(e^{is})$ is absolutely convergent. It is proved in [17] that the singular integral operator S_Γ with the Cauchy kernel acts continuously from $L^p(\Gamma)$ to $L^p(\Gamma)$ $(1 < p < +\infty)$. This is, of course, a very particular case of the famous Calderon theorem, but the simple (but clever) norm estimate of S_Γ given in [17] involves the Wiener-Levy theorem on absolutely convergent Fourier series and can hardly be deduced from general results embracing larger classes of $\Gamma's$.

"Donated" theorems. As mentioned above many of Vinogradov's results are scattered through papers of his friends and colleagues. Always ready to help, he often turned out to be a kind of "mathematical ambulance". Probably S.V. is a Guinness Record Book champion for theorems published in other people's papers. We try to prove this by listing *some* of these theorems.

1) The Vinogradov-Senichkin Test commented on above is first published in [Nik3].

2) There exist $\zeta_n \in \mathbb{T}$ such that $\{r_n\zeta_n : n \geq 1\}$ is a dominating subset of the disk $\mathbb{D}$ if and only if $\sum_n(1 - r_n) = \infty$. First published in [Nik2].

3) Given $a > 0$ and a Carleson sequence $(\lambda_n)_{n\geq 1}$ for the upper half plane $\mathbb{C}_+ = \{z : Im z > 0\}$ such that $lim_n Im(\lambda_n) = \infty$, the exponentials $(e^{i\lambda_n t})$ can be complemented up to a Riesz basis $(e^{i\lambda_n t}) \bigcup (e^{i\lambda'_k t})$ of the space $L^2(0, a)$. First published in [HNP] as Theorem III.3.1. (Later on, this result was considerably developed and applied to frame theory, [S]).

4) Given a Blaschke sequence of radii r_n, i.e., with $\sum_n(1 - r_n) < \infty$, there exist $\zeta_n \in \mathbb{T}$ such that $\{\lambda_n = r_n\zeta_n : n \geq 1\}$ is a strong Carleson sequence $(lim_n|B_{\lambda_n}(\lambda_n)| = 1$ where $B_{\lambda_n} = B/b_{\lambda_n})$. (This strengthens a well known result of Naftalevich). First published in [NikV].

5) There exists a Levin sine-type function F whose zero set Λ is a Carleson sequence in $\mathbb{C}_+$ such that $\sup\{\mathrm{Im}(\lambda) : \lambda \in \Lambda\} = \infty$. First published in [HNP] as Theorem III.3.3 (solving a long standing problem).

6) The description of multA^1_1 mentioned above is first published in [Pe1].

7) A construction of Banach algebras (like $\mathrm{mult}(\mathcal{K}_A) + H^\infty$) needed for resolvent estimates of Toeplitz operators is first published in [Pe2], Section 4, Examples 3 and 4.

8) The Taylor coefficients $\hat{B}(n)$, $n \geq 0$ of a Blaschke product whose zeros form a weak Newman sequence satisfy $\hat{B}(n) = O(1/n)$ for $n \longrightarrow \infty$. First published in [Ver3] where the converse is also proved.

9) One of the main ideas of the paper [TV] by S.Treil and A.Volberg was suggested by Vinogradov according to the authors' acknowledgement (private communication). The idea is that to prove the Muckenhoupt-type weighted estimate for the Hilbert transform one can replace the (previously omnipresent) maximal function techniques by Carleson-type embedding theorems. S.Treil and A.Volberg applied the idea in masterly fashion, finding and proving a matrix valued Hunt-Muckenhoupt-Wheeden theorem, [TV].

10) A non-uniqueness theorem for holomorphic Gevrey classes $G_\alpha = \{f \in Hol(\mathrm{Im}(z) > 0) : |f^{(n)}(z)| \leq C_f Q_f^n n! n^{n/\alpha}\}$: a closed subset $E \subset \mathbb{R}$ whose finite complementary intervals satisfy the following Vinogradov condition (V_α)

$$\sum_k l_k^{1-\alpha} (log^+ \frac{1}{l_k})^{\alpha+\epsilon} < \infty$$

(for some $\epsilon > 0$) is not a uniqueness set for G_α. Published in [Hru] as Theorem 2, this result is still the best known sufficient condition in terms of lengths of the complementary intervals.

* * *

Two adjectives, *elegant* and *exquisite*, come to mind when we try to assess Vinogradov's work as a whole. Elegance is a permanent feature of what he did and how he did it. Exquisiteness marks his very choices of subjects and objects: the space of power series uniformly convergent on $\mathbb{D}$ preferred to the insipid disc algebra and the three parametric Besov scale preferred to Sobolev's, not to mention the dull L^p-scale; he is attracted by mysterious subtleties of multipliers; he tries to divide by an inner factor precisely in the spaces whose resistance to this procedure is most vigorous; not satisfied with the mere interpolation of *values* of a function he wants to simultaneously interpolate its *Fourier coefficients* as well, etc. Such inclinations are not fashionable nowadays, and some colleagues treat them as, at the best, idle manipulations with inequalities, if not corpora delicti. But, even if we forget the intrinsic beauty of Vinogradov's mathematics, such seemingly "exquisite" topics and "technical" tricks are indispensable here and there in Analysis, fashion or no fashion. Powder has to be dry and the weapons sharp and ready.

Stanislav Vinogradov was a faithful votary and skillful master of Analysis.

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